COMPARISON OF VOLUMES OF CONVEX BODIES IN REAL, COMPLEX, AND QUATERNIONIC SPACES

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ABSTRACT. The classical Busemann-Petty problem (1956) asks, whether origin-symmetric convex bodies in $\mathbb{R}^n$ with smaller hyperplane central sections necessarily have smaller volumes. It is known, that the answer is affirmative if $n \leq 4$ and negative if $n > 4$. The same question can be asked when volumes of hyperplane sections are replaced by other comparison functions having geometric meaning. We give unified exposition of this circle of problems in real, complex, and quaternionic n-dimensional spaces. All cases are treated simultaneously. In particular, we show that the Busemann-Petty problem in the quaternionic n-dimensional space has an affirmative answer if and only if $n = 2$. The method relies on the properties of cosine transforms on the unit sphere. Possible generalizations are discussed.

1. INTRODUCTION

Real and complex affine and Euclidean spaces are traditional objects in integral geometry. Similar spaces can be built over more general algebras, in particular, over quaternions. The discovery of quaternions is attributed to W.R. Hamilton (1843).[1] A variety of problems of differential geometry in quaternionic and more general spaces over algebras were investigated by Rosenfel'd and his collaborators, in particular, in the Kasan' geometric school (Russia); see, e.g., [Ros, VSS, Shi]. Some problems of quaternionic integral geometry, mainly related to polytopes and invariant densities, were studied by Coxeter, Cuypers, and others; see [Cu, GNT1, GNT2] and references therein.

In the present article we are focused on comparison problems for convex bodies in the general context of the space $\mathbb{K}^n$, where $\mathbb{K}$ stands for the field $\mathbb{R}$ of real numbers, the field $\mathbb{C}$ of complex numbers, and the

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[1]As is mentioned by Truesdell [Tru, p. 306], "quaternions themselves were first discovered, applied and published by Rodriges, Poisson's former pupil, in 1840".

skew field $\mathbb{H}$ of real quaternions. Since $\mathbb{H}$ is not commutative, special consideration is needed in this case.

Let, for instance, K and L be origin-symmetric convex bodies in $\mathbb{R}^n$ with section functions

$$S_K(H) = vol_{n-1}(K \cap H) \quad \text{and} \quad S_L(H) = vol_{n-1}(L \cap H),$$

H being a hyperplane passing through the origin. Suppose that $S_K(H) \leq S_L(H)$ for all such H. Does it follow that $vol_n(K) \leq vol_n(L)$? Since the latter may not be true, another question arises: For which operator $\mathcal{D}$ is the implication

$$(1.1) \qquad \mathcal{D}S_K(H) \leq \mathcal{D}S_L(H) \quad \forall H \implies vol_n(K) \leq vol_n(L)$$

valid? Comparison problems of this kind attract considerable attention in the last decade, in particular, thanks to remarkable connections with harmonic analysis. The first question is known as the Busemann-Petty (BP) problem [BP]. Many authors contributed to its solution, e.g., Ball [Ba], Barthe, Fradelizi, and Maurey [BFM], Gardner [Ga1, Ga2, GKS], Giannopoulos [Gi], Grinberg and Rivin [GRi], Hadwiger [Ha], Koldobsky [K], Larman and Rogers [LR], Lutwak [Lu], Papadimitrakis [Pa], Rubin [R5], Zhang [Z2]. The answer is really striking. It is "Yes" if and only if $n \leq 4$; see [Ga3, GKS, K, KY], and references therein.

The second question, related to implication (1.1), was asked by Koldobsky, Yaskin, and Yaskina [KYY]. It was called the *modified Busemann-Petty problem*. Both questions were studied by Koldobsky, König, Zymonopoulou [KKZ] and Zymonopoulou [Zy] for convex bodies in $\mathbb{C}^n$. The answer to the first question for $\mathbb{C}^n$ is "Yes" if and only if $n \leq 3$.

We suggest unified exposition of these problems for real, complex, and also quaternionic n-dimensional spaces and the relevant $(n-1)$-dimensional subspaces H. All these cases are treated simultaneously. In particular, we show that *the quaternionic BP problem has an affirmative answer if and only if $n = 2$.*

The article is almost self-contained. Our proofs essentially differ from those in the aforementioned publications and rely on the properties of the generalized cosine transforms on the unit sphere [R2, R3, R7].

The setting of the quaternionic BP problem and its solution require careful preparation and new geometric concepts. The crux is that, unlike the fields of real and complex numbers, the algebra of quaternions is not commutative. This results in non-uniqueness of quaternionic analogues of such concepts as a vector space and its subspaces, a symmetric convex body, a norm, etc.

Another motivation for our work is the lower dimensional Busemann-Petty problem (LDBP), which sounds like the usual BP problem, but the hyperplane sections are replaced by plane sections of fixed dimension $1 < i < n - 1$. In the case $i = 2$, $n = 4$, an affirmative answer to LDBP follows from the solution of the usual BP problem. For $i > 3$, a negative answer was first given by Bourgain and Zhang [BZ]; see also [K, RZ] for alternative proofs. In the cases $i = 2$ and $i = 3$ for $n > 4$, the answer is generally unknown, however, if the body with smaller sections is a body of revolution, the answer is affirmative; see [GZ], [Z1], [RZ]. The paper [R8] contains a solution of the LDBP problem in the more general situation, when the body with smaller sections is invariant under rotations, preserving mutually orthogonal subspaces of dimensions ℓ and $n - \ell$, respectively. The answer essentially depends on ℓ.

It is natural to ask, *how invariance properties of bodies affect the corresponding LDBP problem?*

Of course, this question is too vague, however, every specific example might be of interest. The article [KKZ] on the BP problem in $\mathbb{C}^n$ actually deals with the LDBP problem for $(2n-2)$-dimensional sections of $2n$-dimensional convex bodies, which are invariant under the block diagonal subgroup G of $SO(2n)$ of the form

$$G = \{g = \mathrm{diag}(g_1, \ldots, g_n): \ g_1 = \ldots = g_n \in SO(2)\}.$$

We will show that the BP problem in the n-dimensional left and right quaternionic spaces $\mathbb{H}_l^n$ and $\mathbb{H}_r^n$ is equivalent to the LDBP problem for $(4n - 4)$-dimensional sections of $4n$-dimensional convex bodies, which are invariant under a certain subgroup $G \subset SO(4n)$ of block diagonal matrices, having n equal 4×4 isoclinic (or Clifford) blocks. Every such block is a left (or right) matrix representation of a real quaternion and has the property of rotating all lines through the origin in $\mathbb{R}^4$ by the same angle. We give complete solution to this "G-invariant" comparison problem in the general contest of dn-dimensional convex bodies, $n > 1$, the symmetry of which is determined by complete system of orthonormal tangent vector fields on the unit sphere S^{d-1}. The classical result of differential topology says, that such systems are available only on S^1, S^3, and S^7; see Section 2.5. We also study the corresponding modified BP, when the "derivatives" $\mathcal{D}S_K$ and $\mathcal{D}S_L$ are compared.

Plan of the paper and main results. The significant part of the paper (Sections 2-4) deals with necessary preparations, the aim of which is to make the text accessible to broad audience of analysts and geometers. In Sections 2.1 and 2.2 we recall basic facts about quaternions

and vector spaces $\mathbb{K}^n$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. This information is scattered in the literature; see, e.g., [KS, Lou, Por, Ta, Wo, Z]. We present it in the form, which is suitable for our purposes. Since $\mathbb{H}$ is not commutative, we have to distinguish the left vector space $\mathbb{H}_l^n$ and the right vector space $\mathbb{H}_r^n$.

In Section 2.3 we introduce a concept of *equilibrated body* in the general context of the space $\mathfrak{A}^n$, where $\mathfrak{A}$ is a real associative normed algebra. These bodies serve as a substitute for the class of origin-symmetric convex bodies in $\mathbb{R}^n$. As in the real case (see, e.g., [Bar]), they are associated with norms on $\mathfrak{A}^n$. In the complex case, other names ("absolutely convex" or "balanced") are also in use [GL, Hou, Rob]. We could not find any description of this class of bodies in the quaternionic or more general contexts and present this topic in detail.

In Section 2.4 we give precise setting of the comparison problem of the BP type for equilibrated convex bodies in $\mathbb{K}^n \in \{\mathbb{R}^n, \mathbb{C}^n, \mathbb{H}_l^n, \mathbb{H}_r^n\}$ (see Problem **A**) and the corresponding problems **B** and **C** for G-invariant convex bodies in $\mathbb{R}^N$, $N = dn$. Here $d = 1, 2$, and 4, which corresponds to the real, complex, and quaternionic cases, respectively. Section 2.5 contains necessary information about vector fields on unit spheres and extends problems **B** and **C** to the case $d = 8$. This value of d cannot be increased in the framework of the problems **B** and **C**.

Section 3 provides the reader with necessary background from harmonic analysis related to analytic families of cosine transforms and intersection bodies. The latter were introduced by Lutwak [Lu] and generalized in different directions ; see, e.g, Gardner [Ga3], Goodey, Lutwak, and Weil [GLW], Koldobsky [K], Milman [Mi], Rubin and Zhang [RZ], Zhang [Z1]. Here we follow our previous papers [R2, R3, R7]. We draw attention to Section 3.2 devoted to homogeneous distributions and Riesz fractional derivatives $D^\alpha = (-\Delta)^{\alpha/2}$, where Δ is the Laplace operator on $\mathbb{R}^N$. An important feature of these operators is that the corresponding Fourier multiplier $|y|^\alpha$ does not preserve the Schwartz space $\mathcal{S}(\mathbb{R}^N)$ and the phrases like "in the sense of distributions" (cf. [KYY, KY, Zy]) require careful explanation and justification.

Section 4 is devoted to weighted section functions of origin-symmetric convex bodies. If K is such a body, these functions are defined as i-plane Radon transforms of the characteristic function $\chi_K(x)$, (i.e. $\chi_K(x) = 1$ when $x \in K$, and 0 otherwise) with integration against the weighted Lebesgue measure with a power weight $|x|^\beta$. The usefulness of such functions was first noted in [R4] and mentioned in [RZ, p. 492]. Smoothness properties of these functions play a decisive role in establishing main results, and we study them in detail. Similar properties in the context of the modified BP problem in $\mathbb{R}^n$ and $\mathbb{C}^n$

were briefly indicated in [KYY, KY, Zy], however, the details (which are important and fairly nontrivial) were omitted.

In Section 5 we obtain main results; see Theorems 5.4, 5.5, 5.8, and Corollaries 5.6, 5.7. In particular, *the Busemann-Petty problem in $\mathbb{K}^n$ has an affirmative answer if and only if $n \le 2 + 2/d$, where $d = 1, 2$, and 4 in the real, complex, and quaternionic cases, respectively.*

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Notation. We denote by $\sigma_{n-1} = 2\pi^{n/2}/\Gamma(n/2)$ the area of the unit sphere S^{n-1} in $\mathbb{R}^n$; $SO(n)$ is the special orthogonal group. For $\theta \in S^{n-1}$ and $\gamma \in SO(n)$, $d\theta$ and $d\gamma$ denote the relevant probability measures; $\mathcal{D}(S^{n-1})$ is the space of C^∞-functions on S^{n-1} with standard topology; $\mathcal{D}_e(S^{n-1})$ is the subspace of even functions in $\mathcal{D}(S^{n-1})$.

In the following $M_{n,k}(\mathbb{R})$ is the set of real matrices having n rows and k columns; $M_n(\mathbb{R}) = M_{n,n}(\mathbb{R})$; A^T denotes the transpose of a matrix A; $I_n \in M_n(\mathbb{R})$ is the identity matrix; $V_{n,k} = \{F \in M_{n,k}(\mathbb{R}) : F^T F = I_k\}$ is the Stiefel manifold of orthonormal k-frames in $\mathbb{R}^n$; $\mathrm{Gr}_k(V)$ is the Grassmann manifold of k-dimensional linear subspaces of the vector space V.

Given a certain class X of functions or bodies, we denote by X^G the corresponding subclass of G-invariant objects. For example, $C^G(S^{n-1})$ and $D^G(S^{n-1})$ are the spaces of continuous and infinitely differentiable functions on S^{n-1}, respectively, such that $f(g\theta) = f(\theta) \ \forall g \in G, \ \theta \in S^{n-1}$. An origin-symmetric (o.s.) star body in $\mathbb{R}^n$, $n \ge 2$, is a compact set K with non-empty interior, such that $tK \subset K \ \forall t \in [0, 1]$, $K = -K$, and the *radial function* $\rho_K(\theta) = \sup\{\lambda \ge 0 : \lambda\theta \in K\}$ is continuous on S^{n-1}. We denote by $\mathcal{K}^n$ the set of all o.s. star bodies in $\mathbb{R}^n$. A body $K \in \mathcal{K}^n$ is said to be smooth if $\rho_K \in \mathcal{D}_e(S^{n-1})$.

2. Preliminaries

2.1. **Quaternions.** We regard $\mathbb{H}$ as a normed algebra over $\mathbb{R}$ generated by the units e_0, e_1, e_2, e_3 (the more familiar notation is $\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}$, but we reserve these symbols for other purposes). Every element $q \in \mathbb{H}$ is expressed as $q = q_0 e_0 + q_1 e_1 + q_2 e_2 + q_3 e_3 \ (q_i \in \mathbb{R})$. We set

$$\bar{q} = q_0 e_0 - q_1 e_1 - q_2 e_2 - q_3 e_3, \qquad |q| = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}.$$

The multiplicative structure in $\mathbb{H}$ is governed by the rules

$$e_0 e_i = e_i e_0 = e_i, \qquad i = 0, 1, 2, 3,$$

$$e_1 e_2 = -e_2 e_1 = e_3, \qquad e_2 e_3 = -e_3 e_2 = e_1, \qquad e_3 e_1 = -e_1 e_3 = e_2,$$

$$e_1^2 = e_2^2 = e_3^2 = -e_0.$$

The product of two quaternions $p = p_0e_0 + p_1e_1 + p_2e_2 + p_3e_3$ and $q = q_0e_0 + q_1e_1 + q_2e_2 + q_3e_3$ is computed accordingly as

$$
\begin{aligned}
pq \;=\;& (p_0q_0 - p_1q_1 - p_2q_2 - p_3q_3)e_0 \\
&+\; (p_0q_1 + p_1q_0 + p_2q_3 - p_3q_2)e_1 \\
&+\; (p_0q_2 - p_1q_3 + p_2q_0 + p_3q_1)e_2 \\
&+\; (p_0q_3 + p_1q_2 - p_2q_1 + p_3q_0)e_3,
\end{aligned}
$$

(2.1)

so that

$$
q\bar{q} = \bar{q}q = |q|^2, \quad \overline{pq} = \bar{q}\bar{p}, \quad |pq| = |p||q|, \quad q^{-1} = \bar{q}/|q|^2.
$$

We identify

$$
\mathbb{R} = \{q \in \mathbb{H} : q_1 = q_2 = q_3 = 0\}, \qquad \mathbb{C} = \{q \in \mathbb{H} : q_2 = q_3 = 0\},
$$

and denote by $Sp(1)$ the group of quaternions of absolute value 1. There is a canonical bijection $h : \mathbb{H} \to \mathbb{R}^4$, according to which,

$$
q = q_0e_0 + q_1e_1 + q_2e_2 + q_3e_3 \quad \xrightarrow{\;h\;} \quad v_q = (q_0, q_1, q_2, q_3)^T,
$$

$$
Sp(1) \quad \xrightarrow{\;h\;} \quad S^3.
$$

By (2.1),

$$
\begin{aligned}
p\bar{q} \;=\;& (p_0q_0 + p_1q_1 + p_2q_2 + p_3q_3)e_0 + (-p_0q_1 + p_1q_0 - p_2q_3 + p_3q_2)e_1 \\
&+\; (-p_0q_2 + p_1q_3 + p_2q_0 - p_3q_1)e_2 + (-p_0q_3 - p_1q_2 + p_2q_1 + p_3q_0)e_3,
\end{aligned}
$$

or

$$
(2.2) \qquad\qquad p\bar{q} = \sum_{i=0}^{3} (v_p \cdot A_i v_q)\, e_i,
$$

$$
A_0 = I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad
A_1 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix},
$$

(2.3)

$$
A_2 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \qquad
A_3 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.
$$

Similarly,

$$
(2.4) \qquad\qquad \bar{p}q = \sum_{i=0}^{3} (v_p \cdot A_i' v_q)\, e_i,
$$

$$A_0' = A_0 = I_4, \qquad A_1' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

(2.5)

$$A_2' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \qquad A_3' = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}.$$

One can readily see that $A_i, A_i' \in SO(4)$ and collections

$$\{A_0 v_q, A_1 v_q, A_2 v_q, A_3 v_q\}, \qquad \{A_0' v_q, A_1' v_q, A_2' v_q, A_3' v_q\}$$

form orthonormal bases of $\mathbb{R}^4$ for every $q \in Sp(1)$. This gives the following.

Theorem 2.1. *There exist "left rotations" A_i and "right rotations" A_i' $(i = 1, 2, 3)$, such that for every $\sigma \in S^3$, the frames*

$$\{\sigma, A_1 \sigma, A_2 \sigma, A_3 \sigma\}, \qquad \{\sigma, A_1' \sigma, A_2' \sigma, A_3' \sigma\}$$

form orthonormal bases of $\mathbb{R}^4$.

The left- and right-multiplication mappings $p \to qp$ and $p \to pq$ in $\mathbb{H}$ can be realized as linear transformations of $\mathbb{R}^4$, namely,

(2.6) $$v_{qp} = L_q v_p, \qquad v_{pq} = R_q v_p,$$

(2.7) $$L_q = \begin{bmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & -q_3 & q_2 \\ q_2 & q_3 & q_0 & -q_1 \\ q_3 & -q_2 & q_1 & q_0 \end{bmatrix}, \qquad R_q = \begin{bmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & q_3 & -q_2 \\ q_2 & -q_3 & q_0 & q_1 \\ q_3 & q_2 & -q_1 & q_0 \end{bmatrix}.$$

These formulas define regular representations of $\mathbb{H}$ in the algebra $M_4(\mathbb{R})$ of 4×4 real matrices:

(2.8) $$\rho_l : q \to L_q, \qquad \rho_r : q \to R_{\bar{q}},$$

so that $\rho_l(pq) = \rho_l(p)\rho_l(q)$, $\rho_r(pq) = \rho_r(p)\rho_r(q)$. Clearly,

(2.9) $$L_q = \sum_{i=0}^{3} q_i A_i, \qquad R_{\bar{q}} = \sum_{i=0}^{3} q_i A_i'.$$

In particular,

(2.10) $$A_i = L_{e_i}, \qquad A_i' = R_{\bar{e}_i}, \qquad i = 0, 1, 2, 3.$$

For any $p, q \in \mathbb{H}$, matrices L_p and R_q commute, that is,

(2.11) $$L_p R_q = R_q L_p.$$

Moreover, $\det(L_q) = \det(R_q) = |q|^4$ (see, e.g., [Be, p. 28]). Since the columns of each of these matrices are mutually orthogonal, then, for $|q| = 1$, both matrices belong to $SO(4)$. The map

$$Sp(1) \times Sp(1) \longrightarrow SO(4), \qquad (p, q) \longrightarrow L_p R_{\bar{q}},$$

is a group surjection with kernel $\{(e_0, e_0), \ (-e_0, -e_0)\}$ [Por, Wo]. A direct computation shows that $x \cdot R_q x = x \cdot L_q x = q_0$ for every $x \in S^3$. It means that both L_q and R_q have the property of rotating all half-lines originating from O through the same angle $\cos^{-1} q_0$ (such rotations are called *isoclinic* or *Clifford translations* [Wo]). We call L_q and R_q the *left rotation* and the *right rotation*, respectively. Note also that

$$(2.12) \qquad J L_q J = R_{\bar{q}}, \quad J R_q J = L_{\bar{q}}, \qquad J = \begin{bmatrix} -1 & 0 \\ 0 & I_3 \end{bmatrix}.$$

It means that the left rotation becomes the right one if we change the direction of the first coordinate axis in $\mathbb{R}^4$.

Similarly, if $\mathbb{K} = \mathbb{C}$, we set

$$\mathbb{C} \ni c = a + ib \xrightarrow{h} v_c = (a, b)^T \in \mathbb{R}^2,$$

so that

$$(2.13) \qquad v_{cd} = v_{dc} = M_c v_d; \qquad c, d \in \mathbb{C}, \qquad M_c = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

Clearly, $M_c \in SO(2)$ if $|c| = 1$, and, conversely, every element of $SO(2)$ has the form M_c, $c = \cos \varphi + i \sin \varphi$.

2.2. The space $\mathbb{K}^n$.

Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. Consider the set of "points" $x = (x_1, \dots, x_n)$, $x_i \in \mathbb{K}$, that can be regarded as an additive abelian group in a usual way. We want to equip this set with the structure of the inner product vector space over $\mathbb{K}$. The resulting space will be denoted by $\mathbb{K}^n$. Unlike the cases $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$, in the non-commutative case $\mathbb{K} = \mathbb{H}$ it is necessary to distinguish two types of vector spaces, namely, *right vector spaces* and *left vector spaces*.

We recall (see, e.g., [Art]) that an additive abelian group X is a *right $\mathbb{H}$-vector space* if there is a map $X \times \mathbb{H} \longrightarrow X$, under which the image of each pair $(x, q) \in X \times \mathbb{H}$ is denoted by xq, such that for all $q, q', q'' \in \mathbb{H}$ and $x, x', x'' \in X$,

  (a) $(x' + x'')q = x'q + x''q$;
  (b) $x(q' + q'') = xq' + xq''$;
  (c) $x(q'q'') = (xq')q''$;
  (d) $xe_0 = x$.

Similarly, an additive abelian group X is a *left $\mathbb{H}$-vector space* if there is a map $\mathbb{H} \times X \longrightarrow X$, under which the image of each pair $(q, x) \in \mathbb{H} \times X$ is denoted by qx, such that for all $q, q', q'' \in \mathbb{H}$ and $x, x', x'' \in X$,

(a') $q(x' + x'') = qx' + qx''$;
(b') $(q' + q'')x = q'x + q''x$;
(c') $(q'q'')x = q'(q''x)$;
(d') $e_0 x = x$.

According to these definitions, we define the left vector space $\mathbb{H}_l^n$ to be the space of row vectors $x = (x_1, x_2, \ldots, x_n)$, $x_j \in \mathbb{H}$, with multiplication by scalars $c \in \mathbb{H}$ from the left-hand side $(x \rightarrow cx = (cx_1, cx_2, \ldots, cx_n))$. We equip $\mathbb{H}_l^n$ with the *left inner product*

$$(2.14) \qquad \langle x, y \rangle_l = \sum_{j=1}^{n} x_j \bar{y}_j.$$

The right vector space $\mathbb{H}_r^n$ is defined as the space of column vectors $x = (x_1, x_2, \ldots, x_n)^T$, $x_j \in \mathbb{H}$, with multiplication by scalars $c \in \mathbb{H}$ from the right-hand side $(x \rightarrow xc = (x_1 c, x_2 c, \ldots, x_n c)^T)$ and with the *right inner product*

$$(2.15) \qquad \langle x, y \rangle_r = \sum_{j=1}^{n} \bar{x}_j y_j.$$

Clearly, $\overline{\langle x, y \rangle_l} = \langle y, x \rangle_l$, $\overline{\langle x, y \rangle_r} = \langle y, x \rangle_r$. Furthermore, if $x^* = (\bar{x})^T$, then

$$\langle x, y \rangle_l = \langle x^*, y^* \rangle_r, \qquad \langle x, y \rangle_r = \langle x^*, y^* \rangle_l.$$

If c is a real number, we can write $cx = xc$ for both $x \in \mathbb{H}_l^n$ and $x \in \mathbb{H}_r^n$. If $\mathbb{K} = \mathbb{C}$ (or $\mathbb{R}$) we regard $\mathbb{C}^n$ (or $\mathbb{R}^n$) as the space of column vectors and set

$$(2.16) \qquad \langle x, y \rangle = \sum_{j=1}^{n} \bar{x}_j y_j,$$

as in (2.15) (in the commutative case, definitions (2.14) and (2.15) coincide up to conjugation: $\langle x, y \rangle_l = \overline{\langle x, y \rangle_r}$).

Definition 2.2. *We write $\mathbb{K}^n$ for the vector spaces $\mathbb{H}_l^n$, $\mathbb{H}_r^n$, $\mathbb{C}^n$, and $\mathbb{R}^n$, equipped with the inner product defined above.*

There is a natural bijection $h : \mathbb{K}^n \rightarrow \mathbb{R}^N$, $N = dn$, where $d = 1, 2$, and 4 in the real, complex, and quaternionic case, respectively.

Specifically,

$$(2.17) \qquad \mathbb{H}^n_l \ni x = (x_1, \ldots, x_n) \xrightarrow{\;h\;} v_x = \begin{bmatrix} v_{x_1} \\ \ldots \\ v_{x_n} \end{bmatrix} \in \mathbb{R}^{4n},$$

$$(2.18) \qquad \mathbb{H}^n_r \ni x = \begin{bmatrix} x_1 \\ \ldots \\ x_n \end{bmatrix} \xrightarrow{\;h\;} v_x = \begin{bmatrix} v_{x_1} \\ \ldots \\ v_{x_n} \end{bmatrix} \in \mathbb{R}^{4n},$$

$$(2.19) \qquad \mathbb{C}^n \ni x = (x_1, \ldots, x_n) \xrightarrow{\;h\;} v_x = \begin{bmatrix} v_{x_1} \\ \ldots \\ v_{x_n} \end{bmatrix} \in \mathbb{R}^{2n},$$

where $v_{x_i} = h(x_i)$. Abusing notation, we use the same letter h for both the scalar case, as in Section 2.1, and the vector case, as in (2.17)-(2.19).

Formulas (2.6) and (2.12) have obvious extensions. Namely, for $x \in (\mathbb{H}^n)_l$:

$$(2.20) \quad v_{qx} = \begin{bmatrix} v_{qx_1} \\ \ldots \\ v_{qx_n} \end{bmatrix} = \begin{bmatrix} L_q v_{x_1} \\ \ldots \\ L_q v_{x_n} \end{bmatrix} = Ł_q v_x, \quad Ł_q = \mathrm{diag}(L_q, \ldots, L_q);$$

for $x \in (\mathbb{H}^n)_r$:

$$(2.21) \quad v_{xq} = \begin{bmatrix} v_{x_1 q} \\ \ldots \\ v_{x_n q} \end{bmatrix} = \begin{bmatrix} R_q v_{x_1} \\ \ldots \\ R_q v_{x_n} \end{bmatrix} = \mathcal{R}_q v_x, \quad \mathcal{R}_q = \mathrm{diag}(R_q, \ldots, R_q);$$

$$(2.22) \qquad \mathcal{J} Ł_q \mathcal{J} = \mathcal{R}_{\bar{q}}, \quad \mathcal{J} \mathcal{R}_q \mathcal{J} = Ł_{\bar{q}}, \qquad \mathcal{J} = \mathrm{diag}(J, \ldots, J).$$

Matrices $Ł_q$, $\mathcal{R}_q$, and $\mathcal{J}$ have n blocks; $Ł_q$ and $\mathcal{R}_q$ belong to $SO(4n)$, and $\mathcal{J}^2$ is the identity matrix.

By (2.2), the inner product (2.14) can be written as

$$(2.23) \qquad\qquad \langle x, y \rangle_l = \sum_{i=0}^{3} \langle x, y \rangle_i \, e_i$$

$$(2.24) \quad \langle x, y \rangle_i = v_x \cdot \mathcal{A}_i v_y, \qquad \mathcal{A}_i = \mathrm{diag}(A_i, \ldots, A_i) \quad (n \text{ blocks}),$$

A_i being defined by (2.3). Similarly, by (2.4),

$$(2.25) \qquad\qquad \langle x, y \rangle_r = \sum_{i=0}^{3} \langle x, y \rangle_i' \, e_i,$$

$$(2.26) \qquad \langle x, y \rangle_i' = v_x \cdot \mathcal{A}_i' v_y, \qquad \mathcal{A}_i' = \mathrm{diag}(A_i', \ldots, A_i').$$

By (2.10) and (2.22),

(2.27) $$\mathcal{J}\mathcal{A}_i\mathcal{J} = \mathcal{A}_i', \qquad i = 0, 1, 2, 3.$$

In the case $\mathbb{K} = \mathbb{C}$, for $x \in \mathbb{C}^n$ and $c \in \mathbb{C}$, owing to (2.13), we have

(2.28) $$v_{cx} = v_{xc} = \mathcal{M}_c v_x, \qquad \mathcal{M}_c = \mathrm{diag}(M_c, \ldots, M_c) \in SO(2n).$$

Moreover,

(2.29) $$\langle x, y \rangle = v_x \cdot v_y - i(v_x \cdot \mathcal{B}v_y),$$

(2.30) $$\mathcal{B} = \mathrm{diag}\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right).$$

We introduce the following block diagonal subgroups consisting of n equal isoclinic blocks:

(2.31) $$G_{\mathbb{H},l} = \{g \in SO(4n) :$$
$$g = \mathsf{L}_q = \mathrm{diag}(L_q, \ldots, L_q) \text{ for some } q \in \mathbb{H}, \ |q| = 1\},$$

(2.32) $$G_{\mathbb{H},r} = \{g \in SO(4n) :$$
$$g = \mathcal{R}_q = \mathrm{diag}(R_q, \ldots, R_q) \text{ for some } q \in \mathbb{H}, \ |q| = 1\},$$

(2.33) $$G_{\mathbb{C}} = \{g \in SO(2n) :$$
$$g = \mathcal{M}_c = \mathrm{diag}(M_c, \ldots, M_c) \text{ for some } c \in \mathbb{C}, \ |c| = 1\}.$$

If $\mathbb{K} = \mathbb{R}$, then the corresponding group $G_{\mathbb{R}}$ consists of two elements, namely, I_n and $-I_n$. The groups $G_{\mathbb{H},l}$ and $G_{\mathbb{H},r}$ are conjugate to each other by involution $\mathcal{J}$:

(2.34) $$G_{\mathbb{H},l} = \mathcal{J}G_{\mathbb{H},r}\mathcal{J}.$$

Definition 2.3. *We will use the unified notation G for groups $G_{\mathbb{H},l}$, $G_{\mathbb{H},r}$, $G_{\mathbb{C}}$, and $G_{\mathbb{R}}$.*

2.3. **Equilibrated convex bodies.** It is known that origin-symmetric convex bodies in $\mathbb{R}^n$ are in one-to-one correspondence with norms on $\mathbb{R}^n$. *What is a natural analogue of this class of bodies in spaces over more general fields or algebras?* Below we study this question in the general context of spaces over associative real normed algebras $\mathfrak{A}$ with identity. Our consideration generalizes the known reasoning for real and complex numbers [Bar, GL, Hou, Rob].

We assume that $\mathfrak{A}$ contains real numbers and denote by $|\lambda|$ the norm of an element λ in $\mathfrak{A}$. Let V be a left (or right) module over $\mathfrak{A}$. By relating vectors in V new elements, called points, one obtains an affine space over $\mathfrak{A}$ [Ros]. We keep the same notation V for this affine space. As usual, a set A in V is called convex if $x \in A$ and $y \in A$ implies

$\alpha x + \beta y \in A$ for all $\alpha \geq 0$, $\beta \geq 0$, $\alpha + \beta = 1$. A compact convex set in V with non-empty interior is called a *convex body*.

Definition 2.4. *A set A in a left (right) space V over $\mathfrak{A}$ is called equilibrated if for all $x \in A$, $\lambda x \in A$ ($x\lambda \in A$) whenever $\lambda \in \mathfrak{A}$, $|\lambda| \leq 1$.*

An equilibrated set in $\mathbb{R}^n$ is just an origin-symmetric star-shaped set. The next definition agrees with standard terminology for normed algebras; cf. [Is, p. 655].

Definition 2.5. *Let V be a left space over $\mathfrak{A}$. A function $p : V \to \mathbb{R}$ is called a norm if the following conditions are satisfied:*
   (a) $p(x) \geq 0$ *for all $x \in V$*; $p(x) = 0$ *if and only if $x = 0$;*
   (b) $p(\lambda x) = |\lambda| p(x)$ *for all $x \in V$ and all $\lambda \in \mathfrak{A}$;*
   (c) $p(x + y) \leq p(x) + p(y)$ *for all $x, y \in V$.*
If V is a right space over $\mathfrak{A}$, then (b) *is replaced by*
   (b') $p(x\lambda) = |\lambda| p(x)$ *for all $x \in V$ and all $\lambda \in \mathfrak{A}$.*

Let $V = \mathfrak{A}^n$ be the n-dimensional left (right) affine space over $\mathfrak{A}$. Every point $x \in V$ is represented as $x = x_1 f_1 + \ldots + x_n f_n$ ($x = f_1 x_1 + \ldots + f_n x_n$), where $x_i \in \mathfrak{A}$ and $f_1 = (1, 0, \ldots, 0), \ldots, f_n = (0, 0, \ldots, 1)$ is a standard basis in V. We set $||x||_2 = (\sum_{i=1}^{n} |x_i|^2)^{1/2}$.

Lemma 2.6. *Let $V = \mathfrak{A}^n$ be a left (right) space over $\mathfrak{A}$.*
   (i) *If $p : V \to \mathbb{R}$ is a norm, then*

$$(2.35) \qquad A_p = \{x \in V : p(x) \leq 1\}$$

is an equilibrated convex body.
   (ii) *Conversely, if A is an equilibrated convex body in V, then*

$$(2.36) \qquad p_A(x) = ||x||_A = \inf\{r > 0 : x \in rA\}$$

is a norm in V such that $A = \{x \in V : ||x||_A \leq 1\}$.

The proof of this lemma is standard and is given in Appendix.
In the following $\mathfrak{A} \equiv \mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$; $\mathbb{K}^n$ is any of the spaces $\mathbb{R}^n, \mathbb{C}^n, \mathbb{H}^n_l$ or $\mathbb{H}^n_r$; $G \in \{G_\mathbb{R}, G_\mathbb{C}, G_{\mathbb{H},l}, G_{\mathbb{H},r}\}$; see Definitions 2.2 and 2.3; $N = n, 2n$, or $4n$, respectively. Our next aim is to establish connection between equilibrated convex bodies in $\mathbb{K}^n$ and G-invariant origin-symmetric star bodies in $\mathbb{R}^N = h(\mathbb{K}^n)$. We recall the notation

$$(2.37) \qquad \mathcal{J} = \mathrm{diag}\left(\begin{bmatrix} -1 & 0 \\ 0 & I_3 \end{bmatrix}, \ldots, \begin{bmatrix} -1 & 0 \\ 0 & I_3 \end{bmatrix} \right) \qquad (n \text{ blocks}).$$

Clearly, $\mathcal{J}$ acts on $\xi = (\xi_1, \xi_2, \ldots, \xi_{4n}) \in \mathbb{R}^{4n}$ by converting ξ_1 into $-\xi_1$, ξ_5 into $-\xi_5$, and so on.

Theorem 2.7. *Let A be a set in $\mathbb{K}^n$ and let $B = h(A)$ be its image in $\mathbb{R}^N$. Then*

(i) A is convex if and only if B is convex.

(ii) A is equilibrated in $\mathbb{H}_l^n$ if and only if B is $G_{\mathbb{H},l}$-invariant and star-shaped.

(iii) A is equilibrated in $\mathbb{H}_r^n$ if and only if B is $G_{\mathbb{H},r}$-invariant and star-shaped.

(iv) A is equilibrated in $\mathbb{C}^n$ if and only if B is $G_{\mathbb{C}}$-invariant and star-shaped.

(v) A is equilibrated in $\mathbb{R}^n$ if and only if it is origin-symmetric and star-shaped.

(vi) A set S in $\mathbb{R}^{4n}$ is star-shaped and $G_{\mathbb{H},l}$-invariant (or $G_{\mathbb{H},r}$-invariant) if and only if the reflected set $\mathcal{J}S$ is star-shaped and $G_{\mathbb{H},r}$-invariant ($G_{\mathbb{H},l}$-invariant, respectively).

Proof. (i) Since $h(\alpha x + \beta y) = \alpha h(x) + \beta h(y)$ for all $\alpha, \beta \in \mathbb{R}$ and $x, y \in \mathbb{K}^n$, then A and $B = h(A)$ are convex simultaneously.

(ii) Suppose that $A \subset \mathbb{H}_l^n$ is equilibrated, $\xi \in B$, and $x = h^{-1}(\xi)$. For any $q \in \mathbb{H}$ with $|q| = 1$ we have $qx \in A$, and therefore, $Ł_q \xi = h(qx) \in B$. Furthermore, for any $\lambda \in [0, 1]$, $\lambda \xi = \lambda h(x) = h(\lambda x)$. Since $\lambda x \in A$, then $\lambda \xi \in h(A) = B$. Thus, B is $G_{\mathbb{H},l}$-invariant and star-shaped. Conversely, suppose that $B = h(A)$ is star-shaped and $G_{\mathbb{H},l}$-invariant. Choose any $x \in A$, $q \in \mathbb{H}$, $|q| \leq 1$, and set $q = \lambda \omega$, $\lambda = |q|$, $|\omega| = 1$. We have

$$qx = \lambda \omega x = \lambda h^{-1} h(\omega x) = h^{-1}[\lambda Ł_\omega h(x)].$$

Since $B = h(A)$ is $G_{\mathbb{H},l}$-invariant, then $Ł_\omega h(x) \in B$ and since B is star-shaped, then $\lambda Ł_\omega h(x) \in B$. Hence, $qx = h^{-1}[\lambda Ł_\omega h(x)] \in A$.

The proof of (iii) and (iv) follows the same lines with obvious changes. The statement (v) is trivial. The statement (vi) follows from (2.22). Indeed, let S be a star-shaped $G_{\mathbb{H},l}$-invariant set in $\mathbb{R}^{4n}$ and let $y \in \mathcal{J}S$. Then $y = \mathcal{J}x$, $x \in S$, and for any $q \in \mathbb{H}$ with $|q| = 1$ we have $\mathcal{R}_q y = \mathcal{R}_q \mathcal{J}x = \mathcal{J}\mathcal{J}\mathcal{R}_q \mathcal{J}x = \mathcal{J}Ł_{\bar{q}}x \in \mathcal{J}B$, because $Ł_{\bar{q}}x \in B$. Furthermore, for any $\lambda \in [0, 1]$, $\lambda y = \lambda \mathcal{J}x = \mathcal{J}\lambda x \in \mathcal{J}B$, because $\lambda x \in B$. The reasoning in the opposite direction is similar.   □

2.4. **Central hyperplanes in $\mathbb{K}^n$ and G-invariant Busemann-Petty problem in $\mathbb{R}^N$.** Let $S_{\mathbb{K}^n} = \{y \in \mathbb{K}^n : ||y||_2 = 1\}$ be the unit sphere in $\mathbb{K}^n$. Every hyperplane in $\mathbb{K}^n$ passing through the origin has the form

$$(2.38) \qquad y^\perp = \{x \in \mathbb{K}^n : \langle x, y \rangle = 0\}, \qquad y \in S_{\mathbb{K}^n},$$

where $\langle x, y \rangle$ is the relevant inner product; see (2.14), (2.15), (2.16).

If $\mathbb{K} = \mathbb{R}$, this is a usual $(n-1)$-dimensional subspace of $\mathbb{R}^n$. If $\mathbb{K} = \mathbb{C}$, then, owing to (2.29), the equality $\langle x, y \rangle = 0$ is equivalent to a system of two equations

$$\xi \cdot \theta = 0, \qquad \xi \cdot \mathcal{B}\theta = 0,$$

where $\xi = h(x) \in \mathbb{R}^{2n}$, $\theta = h(y) \in S^{2n-1}$. This system can be replaced by one matrix equation

$$(2.39) \qquad F_2(\theta)^T \xi = 0, \qquad F_2(\theta) = [\theta, \mathcal{B}\theta] \in V_{2n,2},$$

where $V_{2n,2}$ is the Stiefel manifold of orthonormal 2-frames in $\mathbb{R}^{2n}$. Equation (2.39) defines a $(2n-2)$-dimensional subspace of $\mathbb{R}^{2n}$. The collection of all such subspaces will be denoted by $\mathrm{Gr}_{2n-2}^{\mathbb{C}}(\mathbb{R}^{2n})$.

In the non-commutative case $\mathbb{K} = \mathbb{H}$ we have two option. If $\mathbb{K}^n = \mathbb{H}_l^n$, then, owing to (2.23), the equality $\langle x, y \rangle_l = 0$ is equivalent to a system of four equations

$$\xi \cdot \mathcal{A}_i \theta = 0 \qquad (i = 0, 1, 2, 3),$$

or

$$(2.40) \quad F_{4,l}(\theta)^T \xi = 0, \qquad F_{4,l}(\theta) = [\mathcal{A}_0\theta,\ \mathcal{A}_1\theta,\ \mathcal{A}_2\theta,\ \mathcal{A}_3\theta] \in V_{4n,4},$$

where $\xi = h(x) \in \mathbb{R}^{4n}$, and $\theta = h(y) \in S^{4n-1}$ (for simplicity, we use the same letters). If $\mathbb{K}^n = \mathbb{H}_r^n$, then, by (2.25), $\langle x, y \rangle_r = 0$ is equivalent to

$$(2.41) \quad F_{4,r}(\theta)^T \xi = 0, \qquad F_{4,r}(\theta) = [\mathcal{A}_0'\theta,\ \mathcal{A}_1'\theta,\ \mathcal{A}_2'\theta,\ \mathcal{A}_3'\theta] \in V_{4n,4}.$$

Since $\mathcal{A}_i' = \mathcal{J}\mathcal{A}_i\mathcal{J}$ (see (2.27)), then

$$(2.42) \qquad F_{4,r}(\theta) = \mathcal{J}F_{4,l}(\mathcal{J}\theta) \quad \text{for every} \quad \theta \in S^{4n-1}.$$

Thus, (2.40) and (2.41) define two different $(4n-4)$-dimensional subspaces of $\mathbb{R}^{4n}$ generated by the same point $\theta \in S^{4n-1}$. We denote by $\mathrm{Gr}_{4n-4}^{\mathbb{H},l}(\mathbb{R}^{4n})$ and $\mathrm{Gr}_{4n-4}^{\mathbb{H},r}(\mathbb{R}^{4n})$ respective collections of all such subspaces, which are isomorphic to S^{4n-1}. By (2.42),

$$\mathrm{Gr}_{4n-4}^{\mathbb{H},r}(\mathbb{R}^{4n}) = \mathcal{J}\mathrm{Gr}_{4n-4}^{\mathbb{H},l}(\mathbb{R}^{4n}).$$

Given $\theta \in S^{dn-1}$ $(d = 1, 2, 4)$, we will be using the unified notation H_θ for the $(dn - d)$-dimensional subspace orthogonal to $F_1(\theta) = \theta$, $F_2(\theta)$, $F_{4,l}(\theta)$, and $F_{4,r}(\theta)$, respectively.

Proposition 2.8. *The "right" manifold $\mathrm{Gr}_{4n-4}^{\mathbb{H},r}(\mathbb{R}^{4n})$ is invariant under the "left" rotations L_q, that is,*

$$L_q\mathrm{Gr}_{4n-4}^{\mathbb{H},r}(\mathbb{R}^{4n}) = \mathrm{Gr}_{4n-4}^{\mathbb{H},r}(\mathbb{R}^{4n}).$$

The "left" manifold $\mathrm{Gr}_{4n-4}^{\mathbb{H},l}(\mathbb{R}^{4n})$ is invariant under the "right" rotations $\mathcal{R}_q$, that is,

$$\mathcal{R}_q \mathrm{Gr}_{4n-4}^{\mathbb{H},l}(\mathbb{R}^{4n}) = \mathrm{Gr}_{4n-4}^{\mathbb{H},l}(\mathbb{R}^{4n}).$$

Proof. Let $H \in \mathrm{Gr}_{4n-4}^{\mathbb{H},r}(\mathbb{R}^{4n})$, that is, H is orthogonal to $F_{4,r}(\theta) = [\mathcal{A}_0'\theta, \ \mathcal{A}_1'\theta, \ \mathcal{A}_2'\theta, \ \mathcal{A}_3'\theta]$ for some $\theta \in S^{4n-1}$. Since $Ł_p$ and $\mathcal{R}_q$ commute for any $p, q \in \mathbb{H}$ and $\mathcal{A}_i' = \mathcal{R}_{\bar{e}_i}$ (see (2.11) and (2.10)), then $Ł_q \mathcal{A}_i' = \mathcal{A}_i' Ł_q$ and $Ł_q F_{4,r}(\theta) = F_{4,r}(Ł_q \theta)$. This implies

$$Ł_q \mathrm{Gr}_{4n-4}^{\mathbb{H},r}(\mathbb{R}^{4n}) \subset \mathrm{Gr}_{4n-4}^{\mathbb{H},r}(\mathbb{R}^{4n})$$

for the corresponding bundles of subspaces. By the same reason, we have $F_{4,r}(\theta) = Ł_q F_{4,r}(Ł_q^{-1}\theta)$ which gives the opposite embedding. The proof of equality $\mathcal{R}_q \mathrm{Gr}_{4n-4}^{\mathbb{H},l}(\mathbb{R}^{4n}) = \mathrm{Gr}_{4n-4}^{\mathbb{H},l}(\mathbb{R}^{4n})$ is similar. $\square$

The above consideration enables us to give precise setting of the Busemann-Petty problem in $\mathbb{K}^n$ and reformulate the latter as the equivalent lower dimensional problem for G-invariant convex bodies in $\mathbb{R}^N$. We recall that

$$N = dn; \quad n > 1; \quad d = 1, 2, 4; \quad G \in \{G_{\mathbb{R}}, G_{\mathbb{C}}, G_{\mathbb{H},l}, G_{\mathbb{H},r}\};$$

see (2.31) - (2.33). We will be using the unified notation $\tilde{\mathrm{Gr}}_{N-d}(\mathbb{R}^N)$ for the respective manifolds

$$\mathrm{Gr}_{n-1}(\mathbb{R}^n), \quad \mathrm{Gr}_{2n-2}^{\mathbb{C}}(\mathbb{R}^{2n}), \quad \mathrm{Gr}_{4n-4}^{\mathbb{H},l}(\mathbb{R}^{4n}), \quad \mathrm{Gr}_{4n-4}^{\mathbb{H},r}(\mathbb{R}^{4n})$$

of $(N-d)$-dimensional subspaces H_θ introduced above.

Problem A. *Let A and B be equilibrated convex bodies in $\mathbb{K}^n$, $n > 1$, satisfying*

$$(2.43) \qquad\qquad vol_{n-1}(A \cap \xi) \leq vol_{n-1}(B \cap \xi)$$

for all central $\mathbb{K}$-hyperplanes ξ. Does it follow that $vol_n(A) \leq vol_n(B)$?

Here volumes of geometric objects in $\mathbb{K}^n$ are defined as usual volumes of their h-images in $\mathbb{R}^N$, for example,

$$vol_n(A) = vol_N(h(A)), \qquad vol_{n-1}(A \cap \xi) = vol_{N-d}(h(A \cap \xi)).$$

The equivalent lower dimensional problem is formulated as follows.

Problem B. *Let K and L be G-invariant convex bodies in $\mathbb{R}^N$, with section functions*

$$S_K(\theta) = vol_{N-d}(K \cap H_\theta), \qquad S_L(\theta) = vol_{N-d}(L \cap H_\theta),$$

where $H_\theta \in \tilde{\mathrm{Gr}}_{N-d}(\mathbb{R}^N)$. Suppose that $S_K(\theta) \leq S_L(\theta)$ for all $\theta \in S^{N-1}$. Does it follow that $vol_N(K) \leq vol_N(L)$?

We notice a fundamental difference between the usual LDBP problem, where sections by *all* $(N-d)$-dimensional subspaces are compared,

and Problem B, where, in the cases $d = 2$ and 4, the essentially smaller (actually, $(N-1)$-dimensional) collection of subspaces comes into play.

Since the question in Problem **B** may have a negative answer, we also consider the following more general problem, which is of independent interest.

Problem C. *For which operator $\mathcal{D}$ does the assumption $\mathcal{D}S_K(\theta) \leq \mathcal{D}S_L(\theta) \quad \forall \theta \in S^{N-1}$ imply $vol_N(K) \leq vol_N(L)$?*

2.5. **Vector fields on spheres.** Theorem 2.1 suggests intriguing links between possible generalizations of Problems **B** and **C** and the celebrated vector field problem, which asks for the maximal number $\rho(d)$ of orthonormal tangent vector fields on the unit sphere S^{d-1} in $\mathbb{R}^d$.

We recall some facts; see [Hes, Hus, Ad]. A continuous tangent vector field on S^{d-1} is defined to be a continuous function $\mathcal{V} : S^{d-1} \to \mathbb{R}^d$ such that $\mathcal{V}(\sigma) \in \sigma^{\perp}$ for every $\sigma \in S^{d-1}$. If $\mathcal{V}(\sigma) = A\sigma$, where A is a $d \times d$ matrix, the vector field $\mathcal{V}$ is called linear. Vector fields $\mathcal{V}_1, \ldots, \mathcal{V}_k$ on S^{d-1} are called orthonormal if for every $\sigma \in S^{d-1}$, the corresponding vectors $\mathcal{V}_1(\sigma), \ldots, \mathcal{V}_k(\sigma)$ form an orthonormal frame in $\mathbb{R}^d$. The following result is known as the Hurwitz-Radon-Eckmann theorem [Hu, Rad, E]; see also [Og].

Theorem 2.9. *Let d be a positive integer and write $d = 2^{4s+r}t$, where t is an odd integer, s and r are integers with $s \geq 0$ and $0 \leq r < 4$. Then the maximal number of orthonormal linear tangent vector fields on S^{d-1} is equal to $\rho(d) = 2^r + 8s - 1$.*

The number $\rho(d)$ is called the Radon-Hurwitz number. It is zero when d is odd. Adams [Ad] extended this result to continuous vector fields. He proved that there are at most $\rho(d)$ linearly independent continuous tangent vector fields on S^{d-1}.

In the case $\rho(d) = d - 1$, when there exist a complete orthonormal system of linear tangent vector fields $\{\mathcal{V}_1, \ldots, \mathcal{V}_{d-1}\}$ on S^{d-1}, the sphere S^{d-1} is called *parallelizable*. The only parallelizable spheres are S^1, S^3, and S^7; see Kervaire [Ke], Bott and Milnor [BM].

Complete systems of orthonormal linear tangent vector fields on S^3, namely, $\{A_1\sigma, A_2\sigma, A_3\sigma\}$ and $\{A_1'\sigma, A_2'\sigma, A_3'\sigma\}$, where considered in Theorem 2.1. These produce a series of new examples, for instance,

$$(2.44) \qquad \{[\gamma^{-1}A_1\gamma]\sigma, \ [\gamma^{-1}A_2\gamma]\sigma, \ [\gamma^{-1}A_3\gamma]\sigma\}, \qquad \forall \gamma \in O(4).$$

A complete system of orthonormal tangent linear vector fields on S^7 can be constructed, e.g., as follows.

If $\sigma = (\sigma_1,\ \sigma_2,\ \sigma_3,\ \sigma_4,\ \sigma_5,\ \sigma_6,\ \sigma_7,\ \sigma_8)^T \in S^7$, then

$$
\begin{aligned}
A_1\sigma &= (\sigma_2,\ -\sigma_1,\ \sigma_4,\ -\sigma_3,\ \sigma_6,\ -\sigma_5,\ -\sigma_8,\ \sigma_7)^T, \\
A_2\sigma &= (\sigma_3,\ -\sigma_4,\ -\sigma_1,\ \sigma_2,\ \sigma_7,\ \sigma_8,\ -\sigma_5,\ -\sigma_6)^T, \\
A_3\sigma &= (\sigma_4,\ \sigma_3,\ -\sigma_2,\ -\sigma_1,\ \sigma_8,\ -\sigma_7,\ \sigma_6,\ -\sigma_5)^T, \\
A_4\sigma &= (\sigma_5,\ -\sigma_6,\ -\sigma_7,\ -\sigma_8,\ -\sigma_1,\ \sigma_2,\ \sigma_3,\ \sigma_4)^T, \\
A_5\sigma &= (\sigma_6,\ \sigma_5,\ -\sigma_8,\ \sigma_7,\ -\sigma_2,\ -\sigma_1,\ -\sigma_4,\ \sigma_3)^T, \\
A_6\sigma &= (\sigma_7,\ \sigma_8,\ \sigma_5,\ -\sigma_6,\ -\sigma_3,\ \sigma_4,\ -\sigma_1,\ -\sigma_2)^T, \\
A_7\sigma &= (\sigma_8,\ -\sigma_7,\ \sigma_6,\ \sigma_5,\ -\sigma_4,\ -\sigma_3,\ \sigma_2,\ -\sigma_1)^T.
\end{aligned}
$$

The corresponding matrices A_i, which are determined by permutation of indices of coordinates $\sigma_1, \ldots, \sigma_8$ and arrangements of $\pm$ signs, belong to $SO(8)$. More systems can be constructed, e.g., as in (2.44).

The following statement can be found in [Hes] in a slightly more general form. For the sake of completeness, we present it with proof.

Lemma 2.10.
(i) *If $\sigma \to A\sigma$ is a linear tangent vector field on S^{d-1}, then the $d \times d$ matrix A is skew symmetric, that is, $A + A^T = 0$.*
(ii) *If $\mathbf{A}\sigma = \{A_i\sigma\}_{i=1}^{d-1}$ is an orthonormal system of linear tangent vector fields on S^{d-1}, then*

$$A_i^T A_j + A_j^T A_i = 0 \quad \text{for all} \quad 1 \le i < j \le d-1,$$

$$A_i^T A_i = I \quad \text{for all} \quad 1 \le i \le d-1.$$

Proof. (i) Let $\sigma \cdot A\sigma = 0$ for all $\sigma \in S^{d-1}$. Equivalently, $x \cdot Ax = 0$ for all $x \in \mathbb{R}^d$. Then, for all $x, y \in \mathbb{R}^d$,

$$
\begin{aligned}
x \cdot (A + A^T)y &= x \cdot Ay + Ax \cdot y \\
&= x \cdot Ax + x \cdot Ay + Ax \cdot y + Ay \cdot y = (x+y) \cdot A(x+y) = 0.
\end{aligned}
$$

Hence, $A + A^T = 0$.

(ii) As above, for all $x, y \in \mathbb{R}^d$ we have

$$x \cdot (A_i^T A_j + A_j^T A_i)y = A_i(x+y) \cdot A_j(x+y) = 0,$$

$$x \cdot (A_i^T A_i - I)y = \frac{1}{2}\big[A_i(x+y) \cdot A_i(x+y) - (x+y) \cdot (x+y)\big] = 0.$$

This gives the result. $\qquad\square$

Lemma 2.11. *Let $\mathbf{A}\sigma = \{A_i\sigma\}_{i=1}^{d-1}$ be an orthonormal system of linear tangent vector fields on S^{d-1}; $A_0 = I$. Then*

(2.45)
$$g_\lambda(\mathbf{A}) \equiv \sum_{i=0}^{d-1} \lambda_i A_i \in O(d)$$

for every $\lambda = (\lambda_1, \ldots, \lambda_n) \in S^{d-1}$.

Proof. By Lemma 2.10,

$$g_\lambda(\mathbf{A})^T g_\lambda(\mathbf{A}) = \Big(\sum_{i=0}^{d-1} \lambda_i A_i^T \Big)\Big(\sum_{j=0}^{d-1} \lambda_j A_j \Big) = \sum_{i,j=0}^{d-1} \lambda_i \lambda_j A_i^T A_j = I.$$

Hence, $g_\lambda(\mathbf{A}) \in O(d)$. $\square$

Some notation are in order.

Definition 2.12. *Let $N = dn$, $d \in \{2, 4, 8\}$, $n > 1$. Given an orthonormal system $\mathbf{A}\sigma = \{A_i\sigma\}_{i=1}^{d-1}$ of linear tangent vector fields on S^{d-1}, we denote*

$$(2.46)\quad \mathcal{G}_\lambda(\mathbf{A}) = \operatorname{diag}\left(g_\lambda(\mathbf{A}), \dots, g_\lambda(\mathbf{A})\right)$$

$$= \operatorname{diag}\Big(\sum_{i=0}^{d-1} \lambda_i A_i, \dots, \sum_{i=0}^{d-1} \lambda_i A_i \Big) \qquad (n \text{ equal blocks}),$$

where $\lambda \in S^{d-1}$. The corresponding class of block diagonal orthogonal transformations of $\mathbb{R}^N$ (with n equal $d \times d$ diagonal blocks), generated by $\mathbf{A}$, is defined by

$$(2.47)\quad G \equiv G(n, d; \mathbf{A}) = \{g \in O(N) : g = \mathcal{G}_\lambda(\mathbf{A}) \text{ for some } \lambda \in S^{d-1}\}.$$

We also introduce $N \times N$ block diagonal matrices, containing n blocks:

$$(2.48)\qquad \mathcal{A}_i = \operatorname{diag}(A_i, \dots, A_i) \qquad (i = 1, 2, \dots, d-1),$$

and set $\mathcal{A}_0 = I_N$. Given $\theta \in S^{N-1}$, we denote by H_θ the $(N - d)$-dimensional subspace orthogonal to the d-frame

$$(2.49)\qquad\qquad F_d(\theta) = [\theta, \mathcal{A}_1\theta, \dots, \mathcal{A}_{d-1}\theta] \in V_{N,d}$$

and set

$$(2.50)\qquad\qquad \tilde{\mathrm{Gr}}_{N-d}(\mathbb{R}^N) = \{H_\theta : \theta \in S^{N-1}\}.$$

All objects in Definition 2.12 are familiar to us when $d = 2, 4$ (see Section 2.4). Thus, Problems **B** and **C** extend to the case $d = 8$.

We recall that the set G of transformations and the set $\tilde{\mathrm{Gr}}_{N-d}(\mathbb{R}^N)$ of planes are determined by the orthonormal system $\mathbf{A} = \{A_i\}_{i=1}^{d-1}$ of vector fields, which is assumed to be fixed.

The following lemma plays a crucial role in our consideration.

Lemma 2.13. *If $H \in \tilde{\mathrm{Gr}}_{N-d}(\mathbb{R}^N)$, then every continuous G-invariant function f on S^{N-1} is constant on the $(d - 1)$-dimensional section $S^{N-1} \cap H^\perp$.*

Proof. Let $H \equiv H_\theta$ be orthogonal to some d-frame (2.49). Any point $\eta \in S^{N-1} \cap H^\perp$ is represented as

$$\eta = \sum_{i=0}^{d-1} \lambda_i \mathcal{A}_i \theta, \qquad \sum_{i=0}^{d-1} \lambda_i^2 = 1,$$

or $\eta = \mathcal{G}_\lambda(\mathbf{A})\,\theta$; see (2.46). In particular, if $d = 4$ and A_i have the form (2.3), then $\mathcal{G}_\lambda(\mathbf{A})$ is a block diagonal matrix with n equal blocks of the form

$$\sum_{i=0}^{3} \lambda_i A_i = \begin{bmatrix} \lambda_0 & -\lambda_1 & -\lambda_2 & -\lambda_3 \\ \lambda_1 & \lambda_0 & -\lambda_3 & \lambda_2 \\ \lambda_2 & \lambda_3 & \lambda_0 & -\lambda_1 \\ \lambda_3 & -\lambda_2 & \lambda_1 & \lambda_0 \end{bmatrix} = L_\lambda,$$

$$\lambda = \lambda_0 e_0 + \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 \in \mathbb{H}; \quad \text{(cf. (2.7))}.$$

Since $\mathcal{G}_\lambda(\mathbf{A}) \in G$, then $f(\eta) = f(\mathcal{G}_\lambda(\mathbf{A})\,\theta) = f(\theta)$. This gives the result. $\square$

3. Cosine transforms and intersection bodies

It is known [R4, R5, R7, RZ] that diverse Busemann-Petty type problems can be studied using analytic families of cosine transforms on the unit sphere. This approach is parallel, in a sense, to the Fourier transform method developed by Koldobsky and his collaborators [K, KY]. We shall see how these transforms can be applied to Problems **A**, **B**, and **C** stated above.

3.1. **Spherical Radon transforms and cosine transforms.** We recall some basic facts; see [R3, R7]. Fix an integer $i \in \{2, 3, \ldots, N-1\}$ and let $\mathrm{Gr}_i(\mathbb{R}^N)$ be the Grassmann manifold of all i-dimensional linear subspaces ξ of $\mathbb{R}^N$. The spherical Radon transform, that integrates a function $f \in L^1(S^{N-1})$ over $(i-1)$-dimensional sections $S^{N-1} \cap \xi$, is defined by

$$(3.1) \qquad (R_i f)(\xi) = \int_{\theta \in S^{N-1} \cap \xi} f(\theta)\, d_\xi \theta,$$

where $d_\xi \theta$ denotes the probability measures on $S^{N-1} \cap \xi$. The case $i = N - 1$ in (3.1) is known as the Minkowski-Funk transform

$$(3.2) \quad (Mf)(u) = \int_{\{\theta\,:\,\theta \cdot u = 0\}} f(\theta)\, d_u \theta = (R_{N-1} f)(u^\perp), \qquad u \in S^{N-1}.$$

Transformation (3.1) can be regarded as a member (up to a multiplicative constant) of the analytic family of the generalized cosine transforms

$$(3.3) \qquad (R_i^\alpha f)(\xi) = \gamma_{N,i}(\alpha) \int_{S^{N-1}} |\mathrm{Pr}_{\xi^\perp}\theta|^{\alpha+i-N}\, f(\theta)\, d\theta,$$

$$\gamma_{N,i}(\alpha) = \frac{\sigma_{N-1}\,\Gamma((N-\alpha-i)/2)}{2\pi^{(N-1)/2}\,\Gamma(\alpha/2)}, \quad Re\,\alpha > 0, \quad \alpha + i - N \neq 0, 2, 4, \dots .$$

Here $\mathrm{Pr}_{\xi^\perp}\theta$ stands for the orthogonal projection of θ onto $\xi^\perp$. If f is smooth and $Re\,\alpha \leq 0$, then $R_i^\alpha f$ is understood as analytic continuation of integral (3.3), so that

$$(3.4) \qquad \lim_{\alpha \to 0} R_i^\alpha f = R_i^0 f = c_i\, R_i f, \qquad c_i = \frac{\sigma_{i-1}}{2\pi^{(i-1)/2}}.$$

In the case $i = N - 1$ we also set

$$(3.5) \qquad (M^\alpha f)(u) = (R_{N-1}^\alpha f)(u^\perp) = \gamma_N(\alpha) \int_{S^{N-1}} f(\theta)|\theta \cdot u|^{\alpha-1}\, d\theta,$$

$$(3.6) \qquad \gamma_N(\alpha) = \frac{\sigma_{N-1}\,\Gamma\big((1-\alpha)/2\big)}{2\pi^{(N-1)/2}\Gamma(\alpha/2)}, \qquad Re\,\alpha > 0, \quad \alpha \neq 1, 3, 5, \dots .$$

Lemma 3.1. [R7, Lemma 3.2] *Let $\alpha, \beta \in \mathbb{C}$; $\alpha, \beta \neq 1, 3, 5, \dots$. If $\alpha + \beta = 2 - N$ and $f \in \mathcal{D}_e(S^{N-1})$ then*

$$(3.7) \qquad\qquad\qquad M^\alpha M^\beta f = f.$$

If $\alpha, 2-N-\alpha \neq 1, 3, 5, \dots$, then M^α is an automorphism of $\mathcal{D}_e(S^{N-1})$.

Corollary 3.2. *The Minkowski-Funk transform on the space $\mathcal{D}_e(S^{N-1})$ can be inverted by the formula*

$$(3.8) \qquad (M)^{-1} = c_{N-1}\, M^{2-N}, \qquad c_{N-1} = \frac{\sigma_{N-2}}{2\pi^{(N-2)/2}}.$$

Both statements amount to Semyanisty [Se2], who used the Fourier transform techniques. They can also be obtained as immediate consequence of the spherical harmonic decomposition of $M^\alpha f$.

Lemma 3.3. [R7, Lemma 3.5] *Let $Re\,\alpha > 0$; $\alpha \neq 1, 3, 5, \dots$. If $f \in L^1(S^{N-1})$, then*

$$(3.9) \quad (R_i M^\alpha f)(\xi) = c\,(R_{N-i}^{\alpha+i-1} f)(\xi^\perp), \quad \xi \in \mathrm{Gr}_i(\mathbb{R}^N), \quad c = \frac{2\pi^{(i-1)/2}}{\sigma_{i-1}},$$

$$(3.10) \qquad (R_{N-i} M^\alpha f)(\xi^\perp) = \frac{2\pi^{(N-i-1)/2}}{\sigma_{N-i-1}}\,(R_i^{\alpha+N-i-1} f)(\xi).$$

If $f \in \mathcal{D}_e(S^{N-1})$, then (3.9) and (3.10) extend to $Re\,\alpha \leq 0$ by analytic continuation.

Proof. We sketch the proof for the sake of completeness. For $Re\,\alpha > 0$,

$$(R_i M^\alpha f)(\xi) = \gamma_N(\alpha) \int_{S^{N-1}\cap\xi} d_\xi u \int_{S^{N-1}} f(\theta)|\theta \cdot u|^{\alpha-1}\, d\theta.$$

Since $|\theta \cdot u| = |\mathrm{Pr}_\xi \theta||v_\theta \cdot u|$ for some $v_\theta \in S^{N-1} \cap \xi$, changing the order of integration, we obtain

$$(R_i M^\alpha f)(\xi) = \gamma_N(\alpha) \int_{S^{N-1}} f(\theta)|\mathrm{Pr}_\xi \theta|^{\alpha-1}\, d\theta \int_{S^{N-1}\cap\xi} |v_\theta \cdot u|^{\alpha-1} d_\xi u.$$

The inner integral is independent of v_θ and can be easily evaluated. This gives (3.9). Equality (3.10) is a reformulation of (3.9). $\qquad\square$

An origin-symmetric star body K in $\mathbb{R}^N$ is completely determined by its *radial function* $\rho_K(\theta) = \sup\{\lambda \geq 0 : \lambda\theta \in K\}$; see Notation. Passing to polar coordinates, we get

$$(3.11) \qquad \mathrm{vol}_i(K \cap \xi) = \frac{\sigma_{i-1}}{i}(R_i \rho_K^i)(\xi), \qquad \xi \in \mathrm{Gr}_i(\mathbb{R}^N).$$

The next statement follows from Lemma 2.13 and plays the key role in the whole paper.

Lemma 3.4. *Let $\rho_K \in D_e^G(S^{N-1})$, $N = dn$; $d \in \{1, 2, 4, 8\}$, $n > 1$. Then for every subspace $H_\theta \in \tilde{\mathrm{Gr}}_{N-d}(\mathbb{R}^N)$ with $\theta \in S^{N-1}$,*

$$(3.12) \qquad vol_{N-d}(K \cap H_\theta) = \frac{\pi^{N/2-d}\,\sigma_{d-1}}{N-d}(M^{1-d}\rho_K^{N-d})(\theta).$$

Proof. Applying successively (3.11) (with $k = N - d$), (3.4), and (3.10) (with $\alpha = i + 1 - N$, $i = N - d$), we obtain

$$\begin{aligned}
vol_{N-d}(K \cap H_\theta) &= \frac{\sigma_{N-d-1}}{N-d}(R_{N-d}\rho_K^{N-d})(H_\theta) \\
&= \frac{2\pi^{(N-d-1)/2}}{N-d}(R_{N-d}^0 \rho_K^{N-d})(H_\theta) \\
&= \frac{\pi^{N/2-d}\,\sigma_{d-1}}{N-d}(R_d M^{1-d}\rho_K^{N-d})(H_\theta^\perp).
\end{aligned}$$

Since ρ_K is G-invariant and M^{1-d} commutes with orthogonal transformations, then, by Lemma 2.13, $M^{1-d}\rho_K^{N-d} \equiv \mathrm{const}$ on $S^{N-1} \cap H_\theta^\perp$ and (3.12) follows. $\qquad\square$

Remark 3.5. In the classical case $\mathbb{K} = \mathbb{R}$, when $N = n$ and $d = 1$, (3.12) becomes a particular case of (3.11):

$$vol_{n-1}(K \cap \theta^\perp) = \frac{\sigma_{n-2}}{n-1}(M\rho_K^{n-1})(\theta),$$

where M is the Minkowski-Funk transform (3.2).

3.2. **Homogeneous distributions and Riesz fractional derivatives.** Given a G-invariant infinitely smooth body K in $\mathbb{R}^N$ and a plane $H_\theta \in \tilde{\mathrm{Gr}}_{N-d}(\mathbb{R}^N)$ generated by $\theta \in S^{N-1}$, we denote

$$(3.13) \qquad S_K(\theta) = vol_{N-d}(K \cap H_\theta).$$

Question: For which operator A^α,

$$(3.14) \qquad A^\alpha M^{1-d} \rho_K^{N-d} = (M^{1-\alpha} \rho_K^{N-d})(\theta)\,?$$

By (3.12), an answer to this question would give us the corresponding equality for the section function

$$(3.15) \qquad A^\alpha S_K(\theta) = c\,(M^{1-\alpha} \rho_K^{N-d})(\theta), \qquad c = \frac{\pi^{N/2-d}\,\sigma_{d-1}}{N-d},$$

that paves the way to Problem **C**. By Lemma 3.1 we immediately get

$$(3.16) \qquad A^\alpha = M^{1-\alpha} M^{1+d-N}.$$

To make this explicit formula more transparent and convenient to handle, we extend our functions by homogeneity to the entire space $\mathbb{R}^N$ and invoke powers of the Laplacian. This idea was formally used in [KYY, KKZ], but it requires justification and some correction. Below we explain the essence of the matter.

Let $\mathcal{S}(\mathbb{R}^N)$ be the Schwartz space of rapidly decreasing C^∞ functions, and $\mathcal{S}'(\mathbb{R}^N)$ its dual. The Fourier transform of a distribution F in $\mathcal{S}'(\mathbb{R}^N)$ is defined by [2]

$$\langle \hat{F}, \hat{\phi} \rangle = (2\pi)^N \langle F, \phi \rangle, \quad \hat{\phi}(y) = \int_{\mathbb{R}^N} \phi(x)\, e^{ix \cdot y}\, dx, \quad \phi \in \mathcal{S}(\mathbb{R}^N).$$

For $f \in L^1(S^{N-1})$, let

$$(E_\lambda f)(x) = |x|^\lambda f(x/|x|), \qquad x \in \mathbb{R}^N \setminus \{0\}.$$

This operator generates a meromorphic $\mathcal{S}'$-distribution, which is defined by analytic continuation ($a.c.$) as follows:

$$\langle E_\lambda f, \phi \rangle = a.c. \int_0^\infty r^{\lambda+N-1} u(r)\, dr, \quad u(r) = \int_{S^{N-1}} f(\theta)\, \overline{\phi(r\theta)}\, d\theta.$$

The distribution $E_\lambda f$ is regular if $Re\,\lambda > -N$ and admits simple poles at $\lambda = -N, -N-1, \ldots$; see [GS]. If f is orthogonal to all spherical harmonics of degree j, then the derivative $u^{(j)}(r)$ equals zero at $r = 0$ and the pole at $\lambda = -N-j$ is removable. In particular, if f is even, i.e., $(f, \varphi) = (f, \varphi_-)$, $\varphi_-(\theta) = \varphi(-\theta)$ $\forall \varphi \in \mathcal{D}(S^{N-1})$, then the only possible poles of $E_\lambda f$ are $-N, -N-2, -N-4, \ldots$.

---

[2]Here and on, the notation $\langle \cdot, \cdot \rangle$ and $(\cdot, \cdot)$ is used for distributions on $\mathbb{R}^N$ and S^{N-1}, respectively.

Operator family $\{M^\alpha\}$ (see (3.5)) naturally arises thanks to the formula

(3.17) $\qquad [E_{1-N-\alpha}f]^\wedge = 2^{1-\alpha}\pi^{N/2} E_{\alpha-1} M^\alpha f, \qquad f \in \mathcal{D}_e(S^{N-1}),$

which amounts to Semyanistyi [Se2]. It holds pointwise for $0 < Re\,\alpha < 1$ (see, e.g., Lemma 3.3 in [R2]) and extends in the S'-sense to all $\alpha \in \mathbb{C}$ satisfying

(3.18) $\qquad \alpha \notin \{1, 3, 5, \ldots\} \cup \{1 - N, -N - 1, -N - 3, \ldots\}.$

The Riesz fractional derivative $D^\alpha\psi$ of order $\alpha \in \mathbb{C}$ of a Schwartz function ψ is defined as a $\mathcal{S}'(\mathbb{R}^N)$-distribution by the rule

(3.19) $\qquad (2\pi)^N \langle D^\alpha\psi, \phi \rangle = \langle |y|^\alpha \hat{\psi}, \hat{\phi} \rangle, \qquad \phi \in \mathcal{S}(\mathbb{R}^N),$

where the right hand side is a meromorphic function of α with simple poles $\alpha = -N, -N - 2, \ldots$. One can formally regard D^α as a power of minus Laplacian, i.e., $D^\alpha = (-\Delta)^{\alpha/2}$. The case of negative $Re\,\alpha$ corresponds to Riesz potentials [St]. Since multiplication by $|y|^\alpha$ does not preserve the space $\mathcal{S}(\mathbb{R}^N)$, definition (3.19) is not extendable to arbitrary $\mathcal{S}'(\mathbb{R}^N)$-distributions.

To overcome this difficulty, Semyanistyi [Se1] came up with the brilliant idea to introduce another class of distributions as follows. Let $\Psi = \Psi(\mathbb{R}^N)$ be the subspace of $\mathcal{S}(\mathbb{R}^N)$, consisting of functions ω such that $(\partial^\gamma\omega)(0) = 0$ for all multi-indices γ. We denote by $\Phi = \Phi(\mathbb{R}^N)$ the Fourier image of Ψ, which is formed by Schwartz functions orthogonal to all polynomials. Let Φ' and Ψ' be the duals of Φ and Ψ, respectively. Two $\mathcal{S}'$-distributions, that coincide in the Φ'-sense, differ from each other by a polynomial. For any Φ'-distribution g and any $\alpha \in \mathbb{C}$, the Riesz fractional derivative $D^\alpha g$ is correctly defined by the formula

(3.20) $\qquad \langle D^\alpha g, \omega \rangle = (2\pi)^{-N} \langle \hat{g}, |y|^\alpha \hat{\omega} \rangle, \qquad \omega \in \Phi.$

Clearly, multiplication by $|y|^\alpha$ is a linear continuous operator on Ψ (but not on $\mathcal{S}$!); see [R1, SKM] for details and generalizations.

Lemma 3.6. *Let* $\alpha \notin \{0, -2, -4, \ldots\} \cup \{N, N + 2, N + 4, \ldots\}$. *If* $f \in D_e(S^{N-1})$, *then*

(3.21) $\qquad E_{-\alpha} M^{1-\alpha} f = 2^{d-\alpha} D^{\alpha-d} E_{-d} M^{1-d} f$

in the Φ'-*sense. If, moreover,* $\alpha - d = 2m$, $m = 0, 1, 2, \ldots$, *and*

(3.22) $\qquad (D_m f)(\theta) = 2^{-2m} [(-\Delta)^m E_{-d} f](x)|_{x=\theta},$

then

(3.23) $\qquad (M^{1-\alpha} f)(\theta) = (D_m M^{1-d} f)(\theta)$

pointwise for every $\theta \in S^{N-1}$.

Proof. Replace α by $1 - \alpha$ and by $1 - d$ in (3.17). Denoting $c_\alpha = 2^{-\alpha}\pi^{-N/2}$ and $c_d = 2^{-d}\pi^{-N/2}$, we get

$$E_{-\alpha}M^{1-\alpha}f = c_\alpha\,[E_{\alpha-N}f]^\wedge, \qquad E_{-d}M^{1-d}f = c_d\,[E_{d-N}f]^\wedge$$

(in the $\mathcal{S}'$-sense). Using these formulas, for any test function $\omega \in \Phi$ we obtain

$$
\begin{aligned}
\langle E_{-\alpha}M^{1-\alpha}f, \omega \rangle &= c_\alpha\langle [E_{\alpha-N}f]^\wedge, \omega \rangle = c_\alpha\langle E_{\alpha-N}f, \hat{\omega} \rangle \\
&= c_\alpha\langle E_{d-N}f, |y|^{\alpha-d}\hat{\omega} \rangle = c_\alpha\langle E_{d-N}f, [D^{\alpha-d}\omega]^\wedge \rangle \\
&= c_\alpha\langle [E_{d-N}f]^\wedge, D^{\alpha-d}\omega \rangle = c_\alpha c_d^{-1}\langle E_{-d}M^{1-d}f, D^{\alpha-d}\omega \rangle \\
&= 2^{d-\alpha}\langle D^{\alpha-d}E_{-d}M^{1-d}f, \omega \rangle.
\end{aligned}
$$

Let now $\alpha - d = 2m$. Then $D^{\alpha-d} = (-\Delta)^m$, and the same reasoning is applicable for any C^∞-function supported in the neighborhood of the unit sphere. Hence, (3.21) holds pointwise in this specific case, and (3.23) follows. $\square$

Equalities (3.12) and (3.23) imply the following

Corollary 3.7. *Let $S_K(\theta)$, $\theta \in S^{N-1}$, be a section function (3.13) of a G-invariant infinitely smooth body K in $\mathbb{R}^N$; $N = dn$, $n > 1$, $d \in \{1, 2, 4, 8\}$. Let D_m be a differential operator (3.22), where*

$$2m \neq N - d,\ N - d + 2,\ N - d + 4, \dots .$$

Then

$$(3.24) \qquad (D_m S_K)(\theta) = c\,(M^{1-d-2m}\rho_K^{N-d})(\theta), \qquad c = \frac{\pi^{N/2-d}\sigma_{d-1}}{N - d}.$$

3.3. **Intersection bodies.** We recall that $\mathcal{K}^N$ denotes the set of all origin-symmetric star bodies in $\mathbb{R}^N$. According to Lutwak [Lu], a body $K \in \mathcal{K}^N$ is called an intersection body of a body $L \in \mathcal{K}^N$ if $\rho_K(\theta) = \text{vol}_{N-1}(L \cap \theta^\perp)$ for every $\theta \in S^{N-1}$. A wider class of intersection bodies, which is the closure of the Lutwak's class in the radial metric, was introduced by Goodey, Lutwak, and Weil [GLW] as a collection of bodies $K \in \mathcal{K}^N$ with the property $\rho_K = M\mu$, where M is the Minkowski-Funk transform (3.2) and μ is an even nonnegative finite Borel measure on S^{N-1}. The class of all such measures will be denoted by $\mathcal{M}_{e+}(S^{N-1})$.

There exist several generalizations of the concept of intersection body [K, Mi, R7, RZ, Z1]. One of them relies on the fact that the Minkowski-Funk transform M is a member of the analytic family M^α of the cosine transforms.

Definition 3.8. [R7, Definition 5.1] *For $0 < \lambda < N$, a body $K \in \mathcal{K}^N$ is called a λ-intersection body if there is a measure $\mu \in \mathcal{M}_{e+}(S^{N-1})$ such that $\rho_K^\lambda = M^{1-\lambda}\mu$ (by Lemma 3.1, this is equivalent to $M^{1+\lambda-N}\rho_K^\lambda \in \mathcal{M}_{e+}(S^{N-1})$). We denote by $\mathcal{I}_\lambda^N$ the set of all such bodies.*

The equality $\rho_K^\lambda = M^{1-\lambda}\mu$ means that for any $\varphi \in \mathcal{D}(S^{N-1})$,

$$\int_{S^{N-1}} \rho_K^k(\theta)\varphi(\theta)\, d\theta = \int_{S^{N-1}} (M^{1-\lambda}\varphi)(\theta)\, d\mu(\theta),$$

where for $\lambda \geq 1$, $(M^{1-\lambda}\varphi)(\theta)$ is understood in the sense of analytic continuation.[3] If $\lambda = k$ is an integer, the class $\mathcal{I}_\lambda^N$ coincides with Koldobsky's class of k-intersection bodies and agrees with his concept of isometric embedding of the space $(\mathbb{R}^N, ||\cdot||_K)$ into L_{-p}, $p = \lambda$ [K]. In the framework of this concept, all bodies $K \in \mathcal{I}_\lambda^N$ can be regarded as "unit balls of N-dimensional subspaces of $L_{-\lambda}$".

The following statement is a consequence of the trace theorem for cosine transforms; see [R7, Theorem 5.13].

Theorem 3.9. *Let $1 < m < N$, $\eta \in \mathrm{Gr}_m(\mathbb{R}^N)$, and let $0 < \lambda < m$. If $K \in \mathcal{I}_\lambda^N$ in $\mathbb{R}^N$, then $K \cap \eta \in \mathcal{I}_\lambda^m$ in η.*

This fact was used (without proof) in [KKZ, Theorem 4]. In the case, when $\lambda = k$ is an integer, it was established by Milman [Mi]; see [R7, Section 1.1] for the discussion of this statement.

4. Weighted section functions

Let K be an origin-symmetric convex body in $\mathbb{R}^N$. Given a point $z \in int(K)$ (the interior of K), we define the shifted radial function of K with respect to z,

$$(4.1) \quad \rho(z,v) = \sup\{\lambda > 0 : z + \lambda v \in K\}, \quad (z,v) \in \Omega = int(K) \times S^{N-1},$$

which is a distance from z to the boundary of K in the direction v.

Lemma 4.1. [RZ, Lemma 3.1] *If an origin-symmetric convex body K in $\mathbb{R}^N$ has C^m boundary ∂K, $1 \leq m \leq \infty$, then $\rho(z,v) \in C^m(\Omega)$.*

Proof. We recall the proof. Consider the function

$$v = g(z,x) = \frac{x-z}{|x-z|}, \qquad z \in int(K), \ x \in \partial K.$$

Since ∂K is C^m, $g(z,x)$ is a C^m function in $int(K) \times \partial K$. When z is fixed, $g(z,\cdot)$ is a C^m diffeomorphism from ∂K to S^{N-1}. By the

---

[3]There is a typo in [R7]: In Definition 5.1 and in the subsequent equality on p. 712 one should replace ρ_K by ρ_K^λ.

implicit function theorem, $x = f(z, v)$ is a C^m function on Ω. Thus, $\rho(z, v) = |x - z| = |f(z, v) - z|$ is a C^m function on Ω. $\square$

It was discovered by Gardner [Ga1] and Zhang [Z2], that positive solution to the Busemann-Petty problem for convex bodies K in $\mathbb{R}^3$ and $\mathbb{R}^4$ is intimately connected with the volume of parallel hyperplane sections of those bodies; see also [K, KY]. This volume, which is a hyperplane Radon transform of the characteristic function $\chi_K(x)$ of K, is represented as $A_{H,\theta}(t) = vol_{N-1}(K \cap \{H + t\theta\})$, where $t \in \mathbb{R}$, $\theta \in S^{N-1}$, and H is a hyperplane through the origin perpendicular to θ. It was noted in [R4] and in [RZ, p. 492], that further progress can be achieved if we replace $A_{H,\theta}(t)$ by the mean value of the i-plane Radon transform [He, R6] of some weighted function $f(x) = |x|^\beta \chi_K(x)$. This mean value should be taken over all i-planes parallel to a fixed subspace $\xi \in \mathrm{Gr}_i(\mathbb{R}^N)$ at distance $|t|$ from the origin. Such averages for arbitrary f (see [R6, Definition 2.7]) play an important role in the theory of i-plane Radon transforms. Similar "weighted" section functions were later used in [KYY, Zy].

Let us proceed with precise definition. Given a convex body $K \in \mathcal{K}^N$, we define the weighted section function

$$(4.2) \quad A_{i,\beta}(t, \xi) = \int_{S^{N-1} \cap \xi^\perp} \Lambda_\beta(\xi + tu)\, du, \qquad \xi \in \mathrm{Gr}_i(\mathbb{R}^N), \quad t \in \mathbb{R},$$

where

$$(4.3) \qquad\qquad \Lambda_\beta(\xi + tu) = \int_{K \cap (\xi + tu)} |x|^\beta\, dx,$$

is the i-plane Radon transform mentioned above. Clearly, $A_{i,\beta}(t, \xi)$ is an even function of t. Let $B = \{x : |x| \leq 1\}$ be the unit ball in $\mathbb{R}^N$ and let $r_K = \sup\{t > 0 : tB \subset K\}$ be the radius of the inscribed ball in K.

Lemma 4.2. *If a convex body $K \in \mathcal{K}^N$ is infinitely smooth and $\beta > m{-}i$, then all derivatives*

$$A_{i,\beta}^{(j)}(t, \xi) = \left(\frac{d}{dt}\right)^j A_{i,\beta}(t, \xi), \qquad 0 \leq j \leq m,$$

are continuous in $(-r_K, r_K) \times \mathrm{Gr}_i(\mathbb{R}^N)$.

Proof. Passing to polar coordinates in the plane $\xi + tu$, we get

$$(4.4) \qquad\qquad \Lambda_\beta(\xi + tu) = \int_{S^{N-1} \cap \xi} a_{u,v}^\beta(t)\, dv,$$

$$a_{u,v}^\beta(t) = \int_0^{\rho(tu,v)} r^{i-1} (r^2 + t^2)^{\beta/2}\, dr,$$

where $\rho(tu,v)$ is the radial function (4.1). It suffices to show that for $\beta > m-i$, all derivatives $(d/dt)^j a_{u,v}^\beta(t)$, $j = 0, 1, \ldots, m$, are continuous on $(-r_K, r_K)$ uniformly in $(u,v) \in (S^{N-1} \cap \xi^\perp) \times (S^{N-1} \cap \xi)$. Let, for short, $\rho(t) \equiv \rho(tu,v)$. If $m = 0$ and $\beta > -i$ the uniform (in u and v) continuity of $a_{u,v}^\beta(t)$ follows from Lemma 4.1. In the case $m = 1$ we have

$$(d/dt) a_{u,v}^\beta(t) = a_1(t) + a_2(t),$$

where $a_1(t) = \rho^{i-1}(\rho^2 + t^2)^{\beta/2} d\rho/dt$ is nice and $a_2(t) = \beta t a_{u,v}^{\beta-2}(t)$. If $\beta > 2 - i$ we are done. Otherwise, if $1 - i < \beta \le 2 - i$, then

$$(4.5) \quad a_2(t) = \beta t^{i+\beta-1} \int_0^{\rho/t} s^{i-1}(1 + s^2)^{\beta/2-1}\, ds \to 0, \quad \text{as} \quad t \to 0,$$

and the result is still true. Continuing this process, we obtain the required result for all m. $\quad\square$

The next lemma is a slight generalization of the corresponding statements in [KYY] and [Zy].

Lemma 4.3. *Let K be an infinitely smooth origin-symmetric convex body in $\mathbb{R}^N$, $\xi \in \mathrm{Gr}_i(\mathbb{R}^N)$, $1 < i < N$. If $-i < \beta \le 0$, then $A_{i,\beta}(t,\xi) \le A_{i,\beta}(0,\xi)$. If $2 - i < \beta \le 0$, then $(d^2/dt^2) A_{i,\beta}(t,\xi)|_{t=0} \le 0$.*

Proof. Replace $|x|^\beta$ in (4.3) by $-\beta \int_0^{1/|x|} z^{-\beta-1} dz$, $\beta < 0$, and change the order of integration. This gives

$$\Lambda_\beta(\xi + tu) = -\beta \int_0^\infty z^{-\beta-1}\, vol_i((B_{1/z} \cap K) \cap (\xi + tu))\, dz$$

where $B_{1/z}$ is a ball of radius $1/z$ centered at the origin. The integral on the right hand side is well defined if $-i < \beta < 0$. Applying Brunn's theorem to the convex body $B_{1/z} \cap K$, we obtain

$$vol_i((B_{1/z} \cap K) \cap (\xi + tu)) \le vol_i((B_{1/z} \cap K) \cap \xi),$$

which gives the first statement of the lemma. If $2 - i < \beta < 0$, then, by Lemma 4.2, the derivative $(d^2/dt^2) A_{i,\beta}(t,\xi)$ is continuous in the neighborhood of $t = 0$ and the second statement of the lemma follows from the first one. In the case $\beta = 0$ the result follows if we apply Brunn's theorem just to K. $\quad\square$

We recall some facts about analytic continuation (*a.c.*) of integrals

$$(4.6) \qquad I(\alpha) = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} f(t)\, dt, \qquad Re\, \alpha > 0.$$

Lemma 4.4. *Let m be a nonnegative integer, $f \in L^1(\mathbb{R})$.*
(i) *If, moreover, f is m times continuously differentiable in the neighborhood of $t = 0$, then $I(\alpha)$ extends analytically to $\mathrm{Re}\,\alpha > -m$. In particular, for $-m < \mathrm{Re}\,\alpha < -m + 1$,*

$$(4.7) \qquad a.c.\, I(\alpha) = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} \Big[f(t) - \sum_{j=0}^{m-1} \frac{t^j}{j!} f^{(j)}(0) \Big] dt$$

and

$$(4.8) \qquad\qquad \lim_{\alpha \to -m} I(\alpha) = (-1)^m f^{(m)}(0).$$

(ii) *If m is odd and f is an even function, which is $m + 1$ times continuously differentiable in the neighborhood of $t = 0$, then (4.7) holds for $-m - 1 < \mathrm{Re}\,\alpha < -m + 1$.*

Proof. All statements are well known [GS]. For instance, (ii) follows from the fact that all derivatives $f^{(j)}(t)$ of odd order are zero at $t = 0$ and therefore, for m odd, the sum $\sum_{j=0}^{m-1}$ can be replaced by $\sum_{j=0}^{m}$. However, (4.8) is usually proved for functions, which have at least $m+1$ continuous derivatives at $t = 0$. We show that it suffices to have only m continuous derivatives. The latter is important in our consideration. Let

$$\begin{aligned} (I^\lambda f)(t) &= \frac{1}{\Gamma(\lambda)} \int_0^t f(s)(t - s)^{\lambda-1}\, dt \\ &= \frac{t^\lambda}{\Gamma(\lambda)} \int_0^1 f(t\eta)(1 - \eta)^{\lambda-1}\, d\eta, \qquad \lambda > 0, \end{aligned}$$

be the Riemann-Liouville fractional integral of f. Note that

$$f(t) - \sum_{j=0}^{m-1} \frac{t^j}{j!} f^{(j)}(0) = (I^m f^{(m)})(t)$$

and $t^{-m}(I^m f^{(m)})(t) \to f^{(m)}(0)/m!$ as $t \to 0$. Hence, for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that

$$|t^{-m}(I^m f^{(m)})(t) - f^{(m)}(0)/m!| < \varepsilon \qquad \forall t \in (0, \delta).$$

Setting $\alpha = \alpha_0 - m$, $\alpha_0 \in (0, 1)$, we obtain

$$\frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} \Big[f(t) - \sum_{j=0}^{m-1} \frac{t^j}{j!} f^{(j)}(0) \Big] dt - (-1)^m f^{(m)}(0)$$

$$= \frac{1}{\Gamma(\alpha_0 - m)} \int_0^\delta t^{\alpha_0-1} \Big[t^{-m}(I^m f^{(m)})(t) - f^{(m)}(0)/m! \Big] dt$$

$$+ f^{(m)}(0) \Big[\frac{\delta^{\alpha_0}}{\alpha_0\, \Gamma(\alpha_0 - m)\, m!} - (-1)^m \Big]$$

$$+ \frac{1}{\Gamma(\alpha_0 - m)} \int_\delta^\infty t^{\alpha_0-m-1} \Big[f(t) - \sum_{j=0}^{m-1} \frac{t^j}{j!} f^{(j)}(0) \Big] dt = I_1 + I_2 + I_3.$$

If $\alpha_0 \to 0$, then $\alpha_0\, \Gamma(\alpha_0 - m)\, m! \to (-1)^m$,

$$|I_1| < \frac{\varepsilon\, \delta^{\alpha_0}}{\alpha_0\, |\Gamma(\alpha_0 - m)|} \to \varepsilon m!, \qquad I_2 \to 0, \qquad I_3 \to 0.$$

This gives the result. $\hfill \square$

The next lemma establishes connection between weighted section functions, spherical Radon transforms, and cosine transforms.

Lemma 4.5. *Let $\xi \in \mathrm{Gr}_i(\mathbb{R}^N)$, $1 < i < N$. Suppose that*

$$\alpha \neq N - i,\ N - i + 2,\ N - i + 4, \ldots,$$

and K is an infinitely smooth origin-symmetric convex body in $\mathbb{R}^N$.
(i) If $\beta > -i$ and $\mathrm{Re}\,\alpha > 0$, then

(4.9) $\quad \dfrac{1}{\Gamma(\alpha/2)} \displaystyle\int_0^\infty t^{\alpha-1} A_{i,\beta}(t, \xi)\, dt = c\, (R_{N-i} M^{\alpha+1+i-N} \rho_K^{\alpha+\beta+i})(\xi^\perp),$

$$c = \frac{\pi^{i/2}\, \sigma_{N-i-1}}{(\alpha + \beta + i)\, \sigma_{N-1}\, \Gamma((N - i - \alpha)/2)}.$$

(ii) If $\beta > 1 - i$, then (4.9) extends to $-1 < \mathrm{Re}\,\alpha < 0$ as

(4.10) $\qquad \dfrac{1}{\Gamma(\alpha/2)} \displaystyle\int_0^\infty t^{\alpha-1} [A_{i,\beta}(t, \xi) - A_{i,\beta}(0, \xi)]dt$

$$= c\, (R_{N-i} M^{\alpha+1+i-N} \rho_K^{\alpha+\beta+i})(\xi^\perp).$$

(iii) If $\beta \geq 2 - i$, then (4.10) holds in the extended domain $-2 < \mathrm{Re}\,\alpha < 0$.
(iv) If $\beta > m - i$ and $m \geq 0$ is even, then

(4.11) $\quad \dfrac{\Gamma((1 - m)/2)}{2^{m+1}\, \sqrt{\pi}} A_{i,\beta}^{(m)}(0, \xi) = c_1\, (R_{N-i} M^{1-m+i-N} \rho_K^{\beta-m+i})(\xi^\perp),$

$$c_1 = \frac{\pi^{i/2}\, \sigma_{N-i-1}}{(\beta - m + i)\, \sigma_{N-1}\, \Gamma((N - i + m)/2)}.$$

Proof. (i) Consider the integral

$$(4.12) \qquad g_{\alpha,\beta}(\xi) = \frac{1}{\Gamma(\alpha/2)} \int_K |P_{\xi^\perp} x|^{\alpha+i-N} |x|^\beta \, dx, \qquad Re\,\alpha > 0,$$

where $P_{\xi^\perp}$ denotes the orthogonal projection onto $\xi^\perp$. We transform (4.12) in two different ways (a similar trick was used in [R5, p. 61] and [RZ, p. 490]). On the one hand, integration over slices parallel to ξ gives

$$g_{\alpha,\beta}(\xi) = \frac{1}{\Gamma(\alpha/2)} \int_{\xi^\perp} |y|^{\alpha+i-N} \, dy \int_{K\cap(\xi+y)} |x|^\beta \, dx$$

$$(4.13) \qquad\qquad = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty t^{\alpha-1} A_{i,\beta}(t,\xi) \, dt.$$

On the other hand, passing to polar coordinates, we can express $g_{\alpha,\beta}$ as the generalized cosine transform (3.3), namely,

$$g_{\alpha,\beta}(\xi) = \frac{1}{(\alpha+\beta+i)\,\Gamma(\alpha/2)} \int_{S^{N-1}} \rho_K(u)^{\alpha+\beta+i} \, |P_{\xi^\perp}u|^{\alpha+i-N} \, du$$

$$= c_{\alpha,\beta}(R_i^\alpha \rho_K^{\alpha+\beta+i})(\xi),$$

$$c_{\alpha,\beta} = \frac{2\pi^{(N-1)/2}}{(\alpha+\beta+i)\,\sigma_{N-1}\,\Gamma((N-i-\alpha)/2)}.$$

Hence, by (3.9),

$$(4.14) \qquad g_{\alpha,\beta}(\xi) = \frac{c_{\alpha,\beta}\,\sigma_{N-i-1}}{2\pi^{(N-i-1)/2}}\,(R_{N-i}M^{\alpha+1+i-N}\rho_K^{\alpha+\beta+i})(\xi^\perp),$$

which gives (4.9).

(ii) By Lemma 4.2 (with $m = 1$) the derivative $(d/dt)A_{i,\beta}(t,\xi)$ is continuous in the neighborhood of $t = 0$. Keeping in mind that

$$\lim_{\alpha\to-m} \frac{\Gamma(\alpha)}{\Gamma(\alpha/2)} = \frac{\Gamma((1-m)/2)}{2^{m+1}\sqrt{\pi}}$$

and applying Lemma 4.4(i), we obtain (4.10).

(iii) The validity of this statement for $\beta > 2-i$ is a consequence of Lemma 4.2 (with $m = 2$) and Lemma 4.4(ii) (with $m = 1$). Consider the case $\beta = 2-i$ which is more subtle. Denote for short $F(t) = A_{i,\beta}(t,\xi)$ and let first $\beta > 1-i$. By Lemma 4.2 the derivative $F'(t)$ is continuous in the neighborhood of $t = 0$. Since F is an even function, then $F'(0) = 0$ and the left hand side of (4.10) can be written as

$$(4.15) \qquad \frac{1}{\Gamma(\alpha/2)} \int_0^\infty t^{\alpha-1}\Delta(t)\,dt, \quad \Delta(t) = F(t) - F(0) - tF'(0).$$

By (4.2) and (4.4),

$$\Delta(t) = \int_{S^{N-1}\cap\xi^{\perp}} du \int_{S^{N-1}\cap\xi} \Delta_{u,v}(t)dv$$

where $\Delta_{u,v}(t) = f(t) - f(0) - tf'(0)$,

$$f(t) \equiv a_{u,v}^{\beta}(t) = \int_{0}^{\rho(tu,v)} r^{i-1}(r^2 + t^2)^{\beta/2}\,dr, \quad \rho \equiv \rho(tu,v).$$

To estimate $\Delta_{u,v}(t)$, we write it as $\Delta_{u,v}(t) = I_1 + I_2$, where

$$I_1 = \int_{0}^{\rho(tu,v)} r^{i-1}[(r^2 + t^2)^{\beta/2} - r^{\beta}]\,dr,$$

$$I_2 = \int_{0}^{\rho(tu,v)} r^{i+\beta-1}dr - \int_{0}^{\rho(0,v)} r^{i+\beta-1}dr - t[a_1(0) + a_2(0)],$$

$$a_1(t) = \rho^{i-1}(\rho^2 + t^2)^{\beta/2}d\rho/dt, \quad \rho \equiv \rho(tu,v), \quad a_2(t) = \beta t a_{u,v}^{\beta-2}(t).$$

For I_1, changing the order of integration, we have

$$I_1 = \frac{\beta}{2}\int_{0}^{\rho} r^{i-1}dr \int_{0}^{t^2} (r^2 + s)^{\beta/2-1}\,ds = \frac{\beta}{4}\int_{0}^{t^2} s^{(i+\beta)/2-1}\,h(s)ds,$$

$$h(s) = \int_{0}^{\rho^2/s} \eta^{i/2-1}(\eta + 1)^{\beta/2-1}d\eta.$$

If $\beta = 2 - i$ then $h(s) = O(\log(1/s))$ as $s \to 0$ and therefore, $I_1 = O(t^2\log(1/t))$ as $t \to 0$.

To estimate I_2 we note that $a_2(0) = 0$ (see (4.5)) and therefore,

$$\begin{aligned}
I_2 &= \frac{1}{i+\beta}[\rho(tu,v)^{i+\beta} - \rho(0,v)^{i+\beta} - t(i+\beta)\rho(0,v)^{i+\beta-1}\rho'(0,v)] \\
&= \psi(t) - \psi(0) - t\psi'(0), \quad \psi(t) \equiv \rho(tu,v)^{i+\beta}.
\end{aligned}$$

Hence, $I_2 = O(t^2)$ as $t \to 0$. Since all estimates above are uniform in u and v, then the function $\Delta(t)$ in (4.15) is $O(t^2\log(1/t))$ as $t \to 0$. This enables us to extend this integral by analyticity to all $Re\,\alpha > -2$.

The statement (iv) follows from Lemma 4.2 (with $m = 2$) and (4.8).

$$\square$$

5. COMPARISON OF VOLUMES. PROOFS OF THE MAIN RESULTS

We recall basic notation related to Problem B. Let K and L be origin-symmetric convex bodies in $\mathbb{R}^N$, $N = dn$, where $n > 1$, $d \in \{1, 2, 4, 8\}$; G is the class (2.47) of block diagonal orthogonal transformations of $\mathbb{R}^N$, which includes the groups $G_{\mathbb{R}}, G_{\mathbb{C}}, G_{\mathbb{H},l}, G_{\mathbb{H},r}$; see (2.31)-(2.33).

The notation $\tilde{\mathrm{Gr}}_{N-d}(\mathbb{R}^N)$ is used for the respective manifolds (2.50) of $(N-d)$-dimensional subspaces H_θ, $\theta \in S^{N-1}$, in particular, for

$$\mathrm{Gr}_{n-1}(\mathbb{R}^n), \quad \mathrm{Gr}_{2n-2}^{\mathbb{C}}(\mathbb{R}^{2n}), \quad \mathrm{Gr}_{4n-4}^{\mathbb{H},l}(\mathbb{R}^{4n}), \quad \mathrm{Gr}_{4n-4}^{\mathbb{H},r}(\mathbb{R}^{4n});$$

see Section 2.4. If K is an infinitely smooth G-invariant star body in $\mathbb{R}^N$, then, by Lemma 3.4 and Corollary 3.7,

$$(5.1) \qquad S_K(\theta) \equiv vol_{N-d}(K \cap H_\theta) = c\,(M^{1-d}\rho_K^{N-d})(\theta),$$

$$(5.2) \qquad (D_m S_K)(\theta) = c\,(M^{1-d-2m}\rho_K^{N-d})(\theta),$$

where

$$c = \pi^{N/2-d}\sigma_{d-1}/(N-d), \quad (D_m f)(\theta) = 2^{-2m}[(-\Delta)^m E_{-d}f](x)|_{x=\theta},$$

$$(5.3) \qquad 2m \neq N-d, N-d+2, N-d+4, \ldots.$$

Lemma 5.1. *Let*

$$(5.4) \qquad \alpha \notin \{0, -2, -4, \ldots\} \cup \{N, N+2, N+4, \ldots\}.$$

(i) If K and L are infinitely smooth G-invariant star bodies in $\mathbb{R}^N$ such that $(M^{\alpha+1-N}\rho_K^d)(\theta) \geq 0$ and

$$(5.5) \qquad (M^{1-\alpha}\rho_K^{N-d})(\theta) \leq (M^{1-\alpha}\rho_L^{N-d})(\theta) \quad \forall \theta \in S^{N-1},$$

then $vol_N(K) \leq vol_N(L)$.
(ii) If L is an infinitely smooth G-invariant convex body with positive curvature such that $(M^{\alpha+1-N}\rho_L^d)(\theta) < 0$ for some $\theta \in S^{N-1}$, then there exists a G-invariant smooth convex body K for which (5.5) holds, but $vol_N(K) > vol_N(L)$.

Proof. (i) By Lemma 3.1,

$$N\,vol_N(K) = \int_{S^{N-1}} \rho_K^N(\theta)\,d\theta = (\rho_K^{N-d}, \rho_K^d) = (M^{1-\alpha}\rho_K^{N-d}, M^{\alpha+1-N}\rho_K^d).$$

Since $M^{\alpha+1-N}\rho_K^d \geq 0$, we can continue:

$$N\,vol_N(K) \leq (M^{1-\alpha}\rho_L^{N-d}, M^{\alpha+1-N}\rho_K^d) = (\rho_L^{N-d}, \rho_K^d).$$

Now the result follows by Hölder's inequality.

(ii) Let $\varphi(\theta) \equiv (M^{\alpha+1-N}\rho_L^d)(\theta) < 0$ for some $\theta \in S^{N-1}$. Then φ is negative on some open set $\Omega \subset S^{N-1}$ and, by Lemma 3.1, $\rho_L^d = M^{1-\alpha}\varphi$. Since φ is G-invariant, then $\varphi < 0$ on the whole orbit $G\Omega$. Choose a function $\psi \in \mathcal{D}(S^{N-1})$ so that $\psi \neq 0$, $\psi(\theta) > 0$ if $\theta \in G\Omega$, and $\psi(\theta) \equiv 0$ otherwise. Without loss of generality, we can assume ψ to be G-invariant (otherwise, it can be replaced by $\tilde{\psi}(\theta) = \int_G \psi(\gamma\theta)\,d\gamma$). Define a smooth G-invariant body K by $\rho_K^{N-d} = \rho_L^{N-d} - \varepsilon M^{\alpha+1-N}\psi$, $\varepsilon > 0$. If ε is small enough, then K is convex. This conclusion is a

consequence of Oliker's formula [Ol], according to which the Gaussian curvature of an origin-symmetric star body expresses through the first and second derivatives of the radial function. Applying $M^{1-\alpha}$ to the preceding equality, we obtain

$$M^{1-\alpha}\rho_K^{N-d} - M^{1-\alpha}\rho_L^{N-d} = -\varepsilon M^{1-\alpha}M^{\alpha+1-N}\psi = -\varepsilon\psi \leq 0,$$

which gives (5.5). On the other hand,

$$(\rho_L^d, \rho_L^{N-d} - \rho_K^{N-d}) = \varepsilon(M^{1-\alpha}\varphi, M^{\alpha+1-N}\psi) = \varepsilon(\varphi, \psi) < 0$$

or $(\rho_L^d, \rho_L^{N-d}) < (\rho_L^d, \rho_K^{N-d})$. By Hölder's inequality, the latter implies $vol_N(L) < vol_N(K)$. $\qquad\square$

Now, we investigate for which α the inequality $(M^{\alpha+1-N}\rho_K^d)(\theta) \geq 0$ in Lemma 5.1 is available.

Lemma 5.2. *Let K and L be infinitely smooth G-invariant convex bodies in $\mathbb{R}^N$; $N = dn$; $n > 1$; $d \in \{1, 2, 4, 8\}$. Suppose that*

$$(M^{1-\alpha}\rho_K^{N-d})(\theta) \leq (M^{1-\alpha}\rho_L^{N-d})(\theta) \quad \forall\theta \in S^{N-1}$$

for some α satisfying

(5.6) $$\max(N - d - 2, d) \leq \alpha < N.$$

Then $vol_N(K) \leq vol_N(L)$.

Proof. We apply Lemma 4.5 with $\xi = H_\theta$, $i = N - d$, and α replaced by $\alpha + d - N$. By Lemma 2.13 the expression $(R_{N-i}M^{\alpha+1+i-N}\rho_K^{\alpha+\beta+i})(\xi^\perp)$ in Lemma 4.5 transforms into $I_{\alpha,\beta} = (M^{\alpha+1-N}\rho_K^{\alpha+\beta})(\theta)$ and the latter is represented as follows.

• For $\alpha > N - d$, $\beta > d - N$:

(5.7) $$I_{\alpha,\beta} = \frac{c^{-1}}{\Gamma((\alpha + d - N)/2)}\int_0^\infty t^{\alpha+d-N-1}A_{N-d,\beta}(t, H_\theta)\,dt.$$

• For $\alpha = N - d$, $\beta > d - N$:

(5.8) $$I_{\alpha,\beta} = \frac{1}{2}A_{N-d,\beta}(0, H_\theta).$$

• For (a) $N - d - 1 < \alpha < N - d$, $\quad 1 + d - N < \beta \leq 0$, and
$\quad$ (b) $N - d - 2 < \alpha < N - d$, $\quad 2 + d - N \leq \beta \leq 0$:

(5.9) $$I_{\alpha,\beta} = \frac{c^{-1}}{\Gamma((\alpha+d-N)/2)}$$
$$\times \int_0^\infty t^{\alpha+d-N-1}[A_{N-d,\beta}(t, H_\theta) - A_{N-d,\beta}(0, H_\theta)]\,dt.$$

• For $\alpha = N - d - 2, \quad 2 + d - N < \beta \leq 0$:

(5.10)
$$I_{\alpha,\beta} = -\frac{c_1^{-1}}{4} A''_{N-d,\beta}(0, H_\theta).$$

Owing to Lemma 4.3, expressions (5.7)-(5.10) are nonnegative. Set $\beta = d - \alpha$ to get $M^{\alpha+1-N}\rho_K^d \equiv I_{\alpha,d-\alpha}$. Then combine inequalities in each case. We obtain the following bounds for α.

For $d = 1$, $N = n$: $\max(n - 3, 1) \leq \alpha < n$.

For $d = 2, 4, 8$:

   (5.7) holds if $N - d < \alpha < N$.

   (5.8) holds if $\alpha = N - d$.

   (5.9) holds if $N - d - 1 < \alpha < N - d$ when $N \geq 2d + 1$;

                    $N - d - 2 \leq \alpha < N - d$ when $N \geq 2d + 2$;

                            $d \leq \alpha < N - d$ when $2d < N < 2d + 2$.

   (5.10) holds if $\alpha = N - d - 2, \quad N \geq 2d + 2$.

Combining these inequalities, we obtain (5.6).   □

Remark 5.3. Operator $M^{1-\alpha} \equiv (M^{1+\alpha-N})^{-1}$ in Lemmas 5.1 and 5.2, that was originally defined by analytic continuation of the integral (3.5), can be explicitly represented as an integro-differential operator $P(\delta)M^\gamma$, where M^γ, $\gamma > 0$, has the form (3.5) and $P(\delta)$ is a polynomial of the Beltrami-Laplace operator δ on S^{N-1}; see [R2, Section 2.2] for details.

Lemma 5.2 leads to main results of the paper. The next statement gives a positive answer to Problem **B**.

Theorem 5.4. *Let K and L be G-invariant convex bodies in $\mathbb{R}^N$ with section functions*
$$S_K(\theta) = vol_{N-d}(K \cap H_\theta), \qquad S_L(\theta) = vol_{N-d}(L \cap H_\theta),$$
where $H_\theta \in \tilde{\mathrm{Gr}}_{N-d}(\mathbb{R}^N)$, $N = dn$, $n > 1$, $d \in \{1, 2, 4, 8\}$. Suppose that

(5.11)
$$S_K(\theta) \leq S_L(\theta) \qquad \forall \theta \in S^{N-1}.$$

If $n \leq 2 + 2/d$, then $vol_N(K) \leq vol_N(L)$.

Proof. For infinitely smooth bodies the result is contained in Lemma 5.2 (set $\alpha = d$ and make use of (5.1)). Let us extend this result to arbitrary G-invariant convex bodies. Given a G-invariant convex body K, let
$$K^* = \{x : |x \cdot y| \leq 1 \,\forall y \in K\}$$
be the polar body of K with support function
$$h_{K^*}(x) = \max\{x \cdot y : y \in K^*\}.$$

Since $h_{K^*}(\cdot)$ coincides with Minkowski's functional $||\cdot||_K$, then $h_{K^*}(\cdot)$ is G-invariant, and therefore, K^* is G-invariant too. It is known [Schn, pp. 158-161], that any origin-symmetric convex body in $\mathbb{R}^N$ can be approximated by infinitely smooth convex bodies with positive curvature and the approximating operator commutes with rigid motions. Hence, there is a sequence $\{K_j^*\}$ of infinitely smooth G-invariant convex bodies with positive curvature such that $h_{K_j^*}(\theta)$ converges to $h_{K^*}(\theta)$ uniformly on S^{N-1}. The latter means, that for the relevant sequence of infinitely smooth G-invariant convex bodies $K_j = (K_j^*)^*$ we have

$$\lim_{j \to \infty} \max_{\theta \in S^{N-1}} |\, ||\theta||_{K_j} - ||\theta||_K | = 0.$$

This implies convergence in the radial metric, i.e.,

(5.12) $$\lim_{j \to \infty} \max_{\theta \in S^{N-1}} |\rho_{K_j}(\theta) - \rho_K(\theta)| = 0.$$

Let us show that the sequence $\{K_j\}$ in (5.12) can be modified so that $K_j \subset K$. An idea of the argument was borrowed from [RZ]. Without loss of generality, assume that $\rho_K(\theta) \geq 1$. Choose K_j so that

$$|\rho_{K_j}(\theta) - \rho_K(\theta)| < \frac{1}{j+1} \quad \forall \theta \in S^{N-1}$$

and set $K_j' = \frac{j}{j+1} K_j$. Then, obviously, $\rho_{K_j'}(\theta) \to \rho_K(\theta)$ uniformly on S^{N-1} as $j \to \infty$, and

$$\rho_{K_j'} = \frac{j}{j+1} \rho_{K_j} < \frac{j}{j+1}\Big(\rho_K + \frac{1}{j+1}\Big) \leq \rho_K.$$

Hence, $K_j' \subset K$. Now suppose that (5.11) is true. Then it is true when K is replaced by K_j', and, by the assumption of the lemma, $\text{vol}_N(K_j') \leq \text{vol}_N(L)$. Passing to the limit as $j \to \infty$, we obtain $\text{vol}_N(K) \leq \text{vol}_N(L)$. $\square$

The following theorem, which generalizes Theorem 4 from [KKZ], shows that the restriction $n \leq 2 + 2/d$ in Theorem 5.4 is sharp.

Theorem 5.5. *Let $N = dn > 2d + 2$, $n > 1$, $d \in \{1, 2, 4, 8\}$. Then there exist G-invariant infinitely smooth convex bodies K and L in $\mathbb{R}^N$ such that $S_K(\theta) \leq S_L(\theta)$ for all $\theta \in S^{N-1}$, but $\text{vol}_N(K) > \text{vol}_N(L)$.*

Proof. Let $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^N$, $x_j = (x_{j,1} \ldots, x_{j,d})^T$,

$$L = \{x : ||x||_4 = \Big(\sum_{j=1}^{n} |x_j|^4\Big)^{1/4} \leq 1\}.$$

Clearly, L is a G-invariant infinitely smooth convex body. Let X be the $(N - d + 1)$-dimensional subspace of $\mathbb{R}^N$, which consists of vectors

of the form $(x_{1,1}, x_2, \ldots, x_n)^T$. By [K, Theorems 4.19, 4.21], $L \cap X$ is not a λ-intersection body in $\mathbb{R}^{N-d+1}$ if $0 < \lambda < N - d - 2$. Hence, by Theorem 3.9, L is not a λ-intersection body for such λ. It means (see Definition 3.8) that $(M^{1+\lambda-N}\rho_K^\lambda)(\theta) < 0$ for some $\theta \in S^{N-1}$. Set $\lambda = d$ to get $dn > 2d + 2$ and apply Lemma 5.1(ii) with $\alpha = d$. This gives the result. $\square$

Corollary 5.6. *The Busemann-Petty problem* **A** *in* $\mathbb{K}^n$, $n > 1$, *has an affirmative answer if and only if* $n \leq 2 + 2/d$. *In particular,*
   in $\mathbb{R}^n$: *if and only if* $n \leq 4$;
   in $\mathbb{C}^n$: *if and only if* $n \leq 3$;
   in $\mathbb{H}_l^n$ *and* $\mathbb{H}_r^n$: *if and only if* $n = 2$.

Theorem 5.4 also implies the following.

Corollary 5.7. *Let* $d \in \{2, 4, 8\}$, $i = N - d$. *The lower dimensional Busemann-Petty problem for* i-*dimensional sections of* N-*dimensional* G-*invariant convex bodies has an affirmative answer in the following cases:*

$$
\begin{array}{llll}
(a) & N = 4 & (d = 2): & i = 2, \\
(b) & N = 6 & (d = 2): & i = 4, \\
(c) & N = 8 & (d = 4): & i = 4, \\
(d) & N = 10 & (d = 4): & i = 6, \\
(e) & N = 16 & (d = 8): & i = 8.
\end{array}
$$

Another consequence of Lemma 5.2, which addresses Problem **C**, can be obtained if we set $\alpha = d + 2m$ in that Lemma and make use of Corollary 3.7.

Theorem 5.8. *Let* K *and* L *be infinitely smooth* G-*invariant convex bodies in* $\mathbb{R}^N$; $N = dn$, $n > 1$, $d \in \{1, 2, 4, 8\}$. *Suppose that*

$$(-\Delta)^m E_{-d} S_K(\theta) \leq (-\Delta)^m E_{-d} S_L(\theta) \quad \forall \theta \in S^{N-1}$$

for some m *satisfying*

(5.13) $\max(N - 2d - 2, 0) \leq 2m < N - d.$

Then $\mathrm{vol}_N(K) \leq \mathrm{vol}_N(L)$. *In particular,* m *can be chosen as follows:*

For $d = 1$: $m = 0$ *if* $n \leq 4$, *and* $m \in \left\{\dfrac{n-4}{2}, \dfrac{n-3}{2}, \dfrac{n-2}{2}\right\}$ *if* $n > 4$.

For $d = 2$: $m = 0$ *if* $n \leq 3$, *and* $m \in \{n - 3, n - 2\}$ *if* $n > 3$.

For $d = 4$: $m = 0$ *if* $n = 2$, *and* $m \in \{2n - 5, 2n - 4, 2n - 3\}$ *if* $n > 2$.

For $d = 8$: $m = 0$ *if* $n = 2$, *and*

$$m \in \{4n - 9, 4n - 8, 4n - 7, 4n - 6, 4n - 5, \} \text{ if } n > 2.$$

6. APPENDIX: PROOF OF LEMMA 2.6

(i) We recall (see Definition 2.5) that a function $p : V \to \mathbb{R}$ is a norm if the following conditions are satisfied:

(a) $p(x) \geq 0$ for all $x \in V$; $p(x) = 0$ if and only if $x = 0$;

(b) $p(\lambda x) = |\lambda| p(x)$ for all $x \in V$ and all $\lambda \in \mathfrak{A}$;

(c) $p(x + y) \leq p(x) + p(y)$ for all $x, y \in V$.

If V is a right space over $\mathfrak{A}$, then (b) is replaced by

(b') $p(x\lambda) = |\lambda| p(x)$ for all $x \in V$ and all $\lambda \in \mathfrak{A}$.

Let V be a left space (for the right space the argument follows the same lines with (b) replaced by (b')). Suppose that $p : V \to \mathbb{R}$ is a norm and show that

(6.1) $$A_p = \{x \in V : p(x) \leq 1\}$$

is an equilibrated convex body. Let $x, y \in A_p$. Then for any nonnegative α and β satisfying $\alpha + \beta = 1$, owing to (b) and (c), we have

$$p(\alpha x + \beta y) \leq p(\alpha x) + p(\beta y) = \alpha p(x) + \beta p(y) \leq \alpha + \beta = 1.$$

Hence, $\alpha x + \beta y \in A_p$, that is, A_p is convex. Since for every $\lambda \in \mathfrak{A}$ with $|\lambda| \leq 1$, (b) implies $p(\lambda x) = |\lambda| p(x) \leq 1$, then $\lambda x \in A_p$. Thus A_p is equilibrated. To prove that A_p is a body, it suffices to show that A_p is compact and the origin is an interior point of A_p. To this end, we first prove that p is a continuous function. Let $x = x_1 f_1 + \ldots x_n f_n$, as above. By (b) and (c),

$$
\begin{aligned}
p(x) &\leq p(x_1 f_1) + \ldots + p(x_n f_n) = |x_1| p(f_1) + \ldots + |x_n| p(f_n) \\
&\leq \gamma \sum_{j=1}^{n} |x_j|, \qquad \gamma = \max_{j=1,\ldots,n} p(f_j).
\end{aligned}
$$

Now for $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, owing to (c), we have

$$p(x) \leq p(y) + p(x - y), \qquad p(y) \leq p(x) + p(y - x) = p(x) + p(x - y).$$

Hence,

$$|p(x) - p(y)| \leq p(x - y) \leq \gamma \sum_{j=1}^{n} |x_j - y_j|,$$

and the continuity of p follows. Furthermore, since $p(x) > 0$ for every x on the unit sphere $\Omega = \{x \in V : ||x||_2 = 1\}$ and since p is continuous, there exists $\delta > 0$ such that $p(x) > \delta$ for all $x \in \Omega$. If $x \in A_p$ and $x' = x/||x||_2 \in \Omega$, then $1 \geq p(x) = ||x||_2 p(x') > \delta ||x||_2$, i.e., $||x||_2 < \delta^{-1}$. Thus, A_p is bounded. Since A_p is also closed as the inverse image of the closed set $0 \leq \lambda \leq 1$, it is compact.

To prove that A_p is a body, it remains to show that A_p contains the origin in its interior. Since p is continuous and Ω is compact, there is

a number $\beta > 0$ such that $p(x') \leq \beta$ for all $x' \in \Omega$. Then the open ball $B_{1/\beta} = \{x \in V : ||x||_2 < 1/\beta\}$ lies in A_p, because for $x \in B_{1/\beta}$, $p(x) = ||x||_2 p(x') \leq ||x||_2 \beta < 1$.

(ii) Suppose that $A \subset V$ is an equilibrated convex body and let us prove (a)-(c) for $p_A(x) = \inf\{r > 0 : x \in rA\}$. Since A is equilibrated, then $0 \in A$ and therefore, $p_A(0) = \inf\{r > 0 : 0 \in rA\} = 0$. Conversely, if $p_A(x) \equiv \inf\{r > 0 : x \in rA\} = 0$, then for every $k \in \mathbb{N}$, there exists $r_k < 1/k$ such that $x \in r_k A$. Since A is equilibrated, then $r_k A$ is equilibrated too, thanks to the following implications that hold for all $\lambda \in \mathbb{K}$, $|\lambda| \leq 1$:

$$x \in r_k A \Longrightarrow \frac{x}{r_k} \in A \Longrightarrow \frac{\lambda x}{r_k} \in A \Longrightarrow \lambda x \in r_k A.$$

Since $r_k A$ is equilibrated, then $0 \in r_k A$ for all k. Passing in $x \in r_k A$ to the limit as $k \to \infty$, we get $x = 0$. This gives (a).

Let us check (b). For $\lambda = 0$, (b) follows from (a). Let $\lambda \neq 0$. Since A is equilibrated, then for every $r > 0$, $\lambda x \in rA$ if and only if $x \in \frac{r}{|\lambda|}A$. Hence,

$$\begin{aligned} p_A(\lambda x) &= \inf\{r > 0 : \lambda x \in rA\} = \inf\{r > 0 : x \in \frac{r}{|\lambda|}A\} \\ &= |\lambda| \inf\{r > 0 : x \in rA\} = |\lambda| p_A(x). \end{aligned}$$

To prove (c), choose $\alpha, \beta > 0$ and let $x \in \alpha A$, $y \in \beta A$. Then

$$x + y = (\alpha + \beta)\left(\frac{\alpha}{\alpha + \beta}\frac{x}{\alpha} + \frac{\beta}{\alpha + \beta}\frac{y}{\beta}\right).$$

Since the points $\alpha^{-1}x$ and $\beta^{-1}y$ are in A and A is convex, the weighted sum in parentheses is also in A, and therefore, $x + y \in (\alpha + \beta)A$. This gives $p_A(x + y) \leq \alpha + \beta$. By letting $\alpha = p_A(x)$, $\beta = p_A(y)$, we are done. $\qquad\square$

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